

# TRACES ON SYMMETRICALLY NORMED OPERATOR IDEALS

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*We dedicate this paper to the memory of Nigel Kalton whose influence on us both has been profound. Without his collaboration this paper would never have been written.*

**ABSTRACT.** For every symmetrically normed ideal  $\mathcal{E}$  of compact operators, we give a criterion for the existence of a continuous singular trace on  $\mathcal{E}$ . We also give a criterion for the existence of a continuous singular trace on  $\mathcal{E}$  which respects Hardy-Littlewood majorization. We prove that the class of all continuous singular traces on  $\mathcal{E}$  is strictly wider than the class of continuous singular traces which respect Hardy-Littlewood majorization. We establish a canonical bijection between the set of all traces on  $\mathcal{E}$  and the set of all symmetric functionals on the corresponding sequence ideal. Similar results are also proved in the setting of semifinite von Neumann algebras.

## 1. INTRODUCTION

In his groundbreaking paper [6], J. Dixmier proved the existence of positive singular traces (that is, linear positive unitarily invariant functionals which vanish on all finite dimensional operators) on the algebra  $B(H)$  of all bounded linear operators acting on infinite-dimensional separable Hilbert space  $H$ . Namely, if  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a concave increasing function such that

$$(1) \quad \lim_{t \rightarrow \infty} \frac{\psi(2t)}{\psi(t)} = 1,$$

then there is a singular trace  $\tau_\omega$ , defined for every positive compact operator  $A \in B(H)$  by setting

$$(2) \quad \tau_\omega(A) = \omega\left(\frac{1}{\psi(n)} \sum_{k=1}^n s_k(A)\right).$$

Here,  $\{s_k(A)\}_{k \in \mathbb{N}}$  is the sequence of singular values of the compact operator  $A \in B(H)$  taken in the descending order and  $\omega$  is an arbitrary dilation invariant generalised limit on the algebra  $l_\infty$  of all bounded sequences. This trace is finite on  $0 \leq A \in B(H)$  if and only if  $A$  belongs to the Marcinkiewicz ideal (see e.g. [14],[15],[27])

$$\mathcal{M}_\psi := \{A \in B(H) : \sup_{n \in \mathbb{N}} \frac{1}{\psi(n)} \sum_{k=1}^n s_k(A) < \infty\}.$$

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In [18], Dixmier's result was extended to an arbitrary Marcinkiewicz ideal  $\mathcal{M}_\psi$  with the following condition on  $\psi$

$$(3) \quad \liminf_{t \rightarrow \infty} \frac{\psi(2t)}{\psi(t)} = 1.$$

All the traces defined above by formula (2) vanish on the ideal  $\mathcal{L}_1$  consisting of all compact operators  $A \in B(H)$  such that  $\sum_{k=1}^{\infty} s_k(A) < \infty$ .

An ideal  $\mathcal{E}$  of algebra  $B(H)$  is said to be symmetrically normed if  $\{s_k(B)\}_{k \in \mathbb{N}} \leq \{s_k(A)\}_{k \in \mathbb{N}}$  and  $A \in \mathcal{E}$  implies that  $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$  (see [14], [15], [29]<sup>1</sup>, [28], [20]). Since the ideal  $\mathcal{M}_\psi$  is just a special example of symmetrically normed operator ideal, the following question (suggested in [18], [16], [17], [7]) arises naturally.

**Question 1.** *Which symmetrically normed operator ideals admit a nontrivial singular trace<sup>2</sup>?*

In analyzing Dixmier's proof of the linearity of  $\tau_\omega$  given by (1), it was observed in [18] (see also [3]) that  $\tau_\omega$  possesses the following fundamental property, namely if  $0 \leq A, B \in \mathcal{M}_\psi$  are such that

$$(4) \quad \sum_{k=1}^n s_k(B) \leq \sum_{k=1}^n s_k(A), \quad \forall n \in \mathbb{N},$$

then  $\tau_\omega(B) \leq \tau_\omega(A)$ . Such a class of traces was termed “fully symmetric” in [20], [30] (see also earlier papers [8], [25], where the term “symmetric” was used). It is natural to consider such traces only on fully symmetrically normed operator ideals  $\mathcal{E}$  (that is, on symmetrically normed operator ideals  $\mathcal{E}$  satisfying the condition: if  $A, B$  satisfy (4) and  $A \in \mathcal{E}$ , then  $B \in \mathcal{E}$  and  $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$ ). In fact, it was established in [8] that every Marcinkiewicz ideal  $\mathcal{M}_\psi$  with  $\psi$  satisfying the condition (3) possesses fully symmetric traces. Furthermore, in the recent paper [18], the following unexpected result was established. If  $\psi$  satisfies the condition (3), then every fully symmetric trace on  $\mathcal{M}_\psi$  is a Dixmier trace  $\tau_\omega$  for some  $\omega$ .

The following question (also suggested in [18], [7], [16], [17]) arises naturally.

**Question 2.** *Which fully symmetrically normed operator ideals admit a nontrivial singular trace which is fully symmetric?*

In papers [16], [17] the following two problems (closely related to Question 1 and Question 2) were also suggested.

**Question 3.** *Which fully symmetrically normed operator ideals admit a trace which is not fully symmetric?*

Let us fix an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  in  $H$ . An operator  $A \in B(H)$  is called diagonal if  $(Ae_n, e_m) = 0$  for every  $n \neq m$ .

**Question 4.** *Let the mapping  $\varphi : \mathcal{E} \rightarrow \mathbb{C}$  be unitarily invariant. Suppose that  $\varphi$  is linear on the subset of all diagonal operators from  $\mathcal{E}$ . Does it imply that  $\varphi$  is a trace on  $\mathcal{E}$ ?*

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<sup>1</sup> We have to caution the reader that in Theorem 1.16 of [29] the assertion (b) does not hold for the norm of an arbitrary symmetrically normed ideal  $\mathcal{E}$  (see e.g. corresponding counterexamples in [19, p. 83]).

<sup>2</sup> In this paper, we exclusively deal with positive traces

In some very special cases (for principal ideals contained in  $\mathcal{L}_1$ , which are, strictly speaking, not symmetrically normed ideals), Question 3 was answered in the affirmative<sup>3</sup> in [33]. In [20], question 3 was answered in the affirmative for the special case of Marcinkiewicz ideals under the assumption (1). It should be pointed out that the method used in [20] cannot be extended to an arbitrary Marcinkiewicz ideal  $\mathcal{M}_\psi$  and, furthermore, cannot be extended to a general symmetrically normed operator ideal. Question 4 was answered in [20] in full generality using deep results from [11, 10] (see also [9]).

The following theorem is the main result of this paper. It yields answers to Questions 1–3. In the course of the proof of Theorem 5, we also present a new (and very simple) proof answering Question 4. Prior to stating Theorem 5, we make a few preliminary observations, for which we are grateful to the referee.

Any trace  $\varphi : \mathcal{E} \rightarrow \mathbb{C}$  obeys the condition

$$\frac{1}{m}\varphi(A^{\oplus m}) = \varphi(A), \quad A \in \mathcal{E}, m \geq 1.$$

Here, the direct sum  $A^{\oplus m}$  is formed with respect to some arbitrary Hilbert space isomorphism  $H^{\oplus m} \simeq H$ . Thus, traces are closely related to the following convex (see Lemma 11 below) functional on  $\mathcal{E}$ .

$$\pi : A \rightarrow \lim_{m \rightarrow \infty} \frac{1}{m} \|A^{\oplus m}\|_{\mathcal{E}}, \quad A \in \mathcal{E}.$$

The non-triviality of the functional  $\pi : \mathcal{E} \rightarrow \mathbb{R}$  is an obvious necessary condition for the existence of a trace.

**Theorem 5.** *Let  $\mathcal{E}$  be a symmetrically normed operator ideal. Consider the following conditions.*

- (1) *There exist nontrivial singular traces on  $\mathcal{E}$ .*
- (2) *There exist nontrivial singular traces on  $\mathcal{E}$ , which are fully symmetric.*
- (3) *There exist nontrivial singular traces on  $\mathcal{E}$ , which are not fully symmetric.*
- (4)  *$\mathcal{E} \neq \mathcal{L}_1$  and there exist an operator  $A \in \mathcal{E}$  such that*

$$(5) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \|A^{\oplus m}\|_{\mathcal{E}} > 0.$$

- (i) *The conditions (1) and (4) are equivalent for every symmetrically normed operator ideal  $\mathcal{E}$ .*
- (ii) *The conditions (1), (2) and (4) are equivalent for every fully symmetrically normed operator ideal  $\mathcal{E}$ .*
- (iii) *The conditions (1) – (4) are equivalent for every fully symmetrically normed operator ideal  $\mathcal{E}$  equipped with a Fatou norm.*

Recall that the norm on a symmetrically normed operator ideal  $\mathcal{E}$  is called a Fatou norm if the unit ball of  $\mathcal{E}$  is closed with respect to strong (or, equivalently, weak) operator convergence. Observe that classical ideals (such as Schatten-von Neumann ideals  $\mathcal{L}_p$ , Marcinkiewicz, Orlicz and Lorentz ideals [14], [15], [29]) have a Fatou norm. In fact, in some standard references on the subject (e.g. Simon's book [29]), the requirement that symmetrically normed operator ideal has a Fatou norm appears to be a part of the definition. Similarly, in the book [24], devoted to the study of symmetric<sup>4</sup> function spaces (which are a commutative counterpart

<sup>3</sup>We are grateful to the referee for this remark.

<sup>4</sup>termed there “rearrangement invariant”.

of symmetrically normed operator ideals), an assumption that the norm is a Fatou norm is incorporated into the definition [24, p. 118].

The proof of Theorem 5 is given in Section 7. In fact, in this paper we will prove a more general result for symmetric spaces associated with semifinite von Neumann algebras. The precise statements are given in Section 4 (see Theorems 23, 28, 29), Section 5 (see Theorems 33, 35, 36) and Section 6 (see Theorems 47, 48). The appendix contains the proof of important technical results for which we were unable to find a suitable reference. We also present a new and short proof of the Figiel-Kalton theorem from [13].

Finally, we say a few words about our proof and its relation to the previous results in the literature. Our strategy is based on the approach from recent papers [30] and [21], where condition (5) was connected to the geometry of  $\mathcal{E}$  (see also [2]). The condition (5) is easy to verify in concrete situations. For example, the following corollary of Theorem 5 strengthens the main result of [20] and complements earlier results of J. Varga [32].

**Corollary 6.** *Every Marcinkiewicz ideal  $\mathcal{M}_\psi$  with  $\psi$  satisfying the condition (3) admits a trace which is not fully symmetric.*

Indeed, it is proved in [1, Proposition 2.3] that the condition (4) of Theorem 5 is equivalent to the condition (3) for the Marcinkiewicz ideal  $\mathcal{M}_\psi$ . Some examples of symmetrically normed operator ideals, which are not Marcinkiewicz ideals, possessing symmetric traces were presented in [7]. These results are also an immediate corollary of Theorem 5.

For completeness, we note that the assertion (ii) in Theorem 5 holds for a wider class of relatively fully symmetrically normed operator ideals. The latter class is defined as follows: if  $A, B \in \mathcal{E}$  are such that (4) holds, then  $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$ . It coincides with the class of all symmetrically normed subspaces of a fully symmetric operator ideal (see [19])

## 2. DEFINITIONS AND PRELIMINARIES

The theory of singular traces on symmetric operator ideals rests on some classical analysis which we now review for completeness.

As usual,  $L_\infty(0, \infty)$  is the set of all bounded Lebesgue measurable functions on the semi-axis equipped with the uniform norm. Given a function  $x \in L_\infty(0, \infty)$ , one defines its decreasing rearrangement  $t \rightarrow \mu(t, x)$  by the formula (see e.g. [22])

$$\mu(t, x) = \inf\{s \geq 0 : m(\{x > s\}) \leq t\}.$$

Let  $H$  be a Hilbert space and let  $B(H)$  be the algebra of all bounded operators on  $H$  equipped with the uniform norm.

Let  $\mathcal{M} \subset B(H)$  be a semi-finite von Neumann algebra equipped with a fixed faithful and normal semi-finite trace  $\tau$ .  $\mathcal{M}$  is said to be atomic (see [31, Definition 5.9]) if every nonzero projection in  $\mathcal{M}$  contains a nonzero minimal projection.  $\mathcal{M}$  is said to be atomless if there is no minimal projections in  $\mathcal{M}$ .

For every  $A \in \mathcal{M}$ , the generalised singular value function  $t \rightarrow \mu(t, A)$  is defined by the formula (see e.g. [12])

$$\mu(t, A) = \inf\{\|Ap\| : \tau(1 - p) \leq t\}.$$

If, in particular,  $\mathcal{M} = B(H)$ , then  $\mu(A)$  is a step function and, therefore, can be identified with the sequence of singular numbers of the operators  $A$  (the singular

values are the eigenvalues of the operator  $|A| = (A^*A)^{1/2}$  arranged with multiplicity in decreasing order).

Equivalently,  $\mu(A)$  can be defined in terms of the distribution function  $d_A$  of  $A$ . That is, setting

$$d_A(s) = \tau(E^{|A|}(s, \infty)), \quad s \geq 0,$$

we obtain

$$\mu(t, A) = \inf\{s \geq 0 : d_A(s) \leq t\}, \quad t > 0.$$

Here,  $E^{|A|}$  denotes the spectral measure of the operator  $|A|$ .

Using the Jordan decomposition, every operator  $A \in B(H)$  can be uniquely written as

$$A = (\Re A)_+ - (\Re A)_- + i(\Im A)_+ - i(\Im A)_-.$$

Here,  $\Re A := 1/2(A + A^*)$  (respectively,  $\Im A := 1/2i(A - A^*)$ ) for any operator  $A \in B(H)$  and  $B_+ = BE^B(0, \infty)$  (respectively,  $B_- = -BE^B(-\infty, 0)$ ) for any self-adjoint operator  $B \in B(H)$ . Recall that  $\Re A, \Im A \in \mathcal{M}$  for every  $A \in \mathcal{M}$  and  $B_+, B_- \in \mathcal{M}$  for every self-adjoint  $B \in \mathcal{M}$ .

Further, we need to recall the important notion of Hardy–Littlewood majorization. Let  $A, B \in (L_1 + L_\infty)(\mathcal{M})$ . The operator  $B$  is said to be majorized by  $A$  and written  $B \prec\prec A$  if and only if

$$\int_0^t \mu(s, B) ds \leq \int_0^t \mu(s, A) ds, \quad t \geq 0.$$

We have (see [12])

$$A + B \prec\prec \mu(A) + \mu(B) \prec\prec 2\sigma_{1/2}\mu(A + B)$$

for every positive operators  $A, B \in (L_1 + L_\infty)(\mathcal{M})$ .

If  $s > 0$ , the dilation operator  $\sigma_s$  is defined by setting

$$(\sigma_s(x))(t) = x\left(\frac{t}{s}\right), \quad t > 0$$

in the case of the semi-axis. In the case of the interval  $(0, 1)$ , the operator  $\sigma_s$  is defined by

$$(\sigma_s x)(t) = \begin{cases} x(t/s), & t \leq \min\{1, s\} \\ 0, & s < t \leq 1. \end{cases}$$

Similarly, in the sequence case, we define an operator  $\sigma_n$  by setting

$$\sigma_n(a_1, a_2, \dots) = (\underbrace{a_1, \dots, a_1}_{n \text{ times}}, \underbrace{a_2, \dots, a_2}_{n \text{ times}}, \dots)$$

and an operator  $\sigma_{1/2}$  by setting

$$\sigma_{1/2} : (a_1, a_2, a_3, a_4, \dots) \rightarrow \left(\frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}, \dots\right).$$

**Definition 7.** The Banach space  $E(\mathcal{M}, \tau) \subset (L_1 + L_\infty)(\mathcal{M})$  is said to be a symmetric operator space if the following conditions hold.

- (1) Given  $A \in E(\mathcal{M}, \tau)$  and  $B \in (L_1 + L_\infty)(\mathcal{M})$  with  $\mu(B) = \mu(A)$ , we have  $B \in E(\mathcal{M}, \tau)$  and  $\|B\|_E = \|A\|_E$ .
- (2) Given  $0 \leq A \in E(\mathcal{M}, \tau)$  and  $0 \leq B \in (L_1 + L_\infty)(\mathcal{M})$  with  $B \leq A$ , we have  $B \in E(\mathcal{M}, \tau)$  and  $\|B\|_E \leq \|A\|_E$ .

The space  $E(\mathcal{M}, \tau)$  is called fully symmetric if for every  $A \in E(\mathcal{M}, \tau)$  and every  $B \in (L_1 + L_\infty)(\mathcal{M})$  with  $B \prec\prec A$ , we have  $B \in E(\mathcal{M}, \tau)$  and  $\|B\|_E \leq \|A\|_E$ .

The norm on a symmetric space  $E(\mathcal{M}, \tau)$  is a Fatou norm if the unit ball of  $E(\mathcal{M}, \tau)$  is closed with respect to strong (or, equivalently, weak) operator convergence. Every symmetric space equipped with a Fatou norm is necessarily fully symmetric.

A linear functional  $\varphi : E(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  is said to be symmetric if  $\varphi(B) = \varphi(A)$  for every positive  $A, B \in E(\mathcal{M}, \tau)$  such that  $\mu(B) = \mu(A)$ . A linear functional  $\varphi : E(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  is said to be fully symmetric if  $\varphi(B) \leq \varphi(A)$  for every positive  $A, B \in E(\mathcal{M}, \tau)$  such that  $B \prec\prec A$ . Every fully symmetric functional is symmetric and bounded. The converse fails [20].

A functional  $\varphi : E(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  is called singular if  $\varphi = 0$  on  $(L_1 \cap L_\infty)(\mathcal{M})$ . If  $E(\mathcal{M}, \tau) \not\subset L_1(\mathcal{M})$ , then every symmetric functional is singular.

If  $E = E(0, \infty)$  and if  $\varphi : E \rightarrow \mathbb{R}$  is a symmetric functional, then  $s\varphi(x) = \varphi(\sigma_s x)$  for every  $x \in E$ . If  $E = E(0, 1)$  and if  $\varphi : E \rightarrow \mathbb{R}$  is a singular symmetric functional, then  $s\varphi(x) = \varphi(\sigma_s x)$  for every  $x = \mu(x) \in E$ .

Let  $E$  be a fully symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. We need the notion of an expectation operator (see [2]).

Let  $\mathcal{A} = \{A_k\}$  be a (finite or infinite) sequence of disjoint sets of finite measure and denote by  $\mathfrak{A}$  the collection of all such sequences. Denote by  $A_\infty$  the complement of  $\cup_k A_k$ .

The expectation operator  $\mathbf{E}(\cdot|\mathcal{A}) : L_1 + L_\infty \rightarrow L_1 + L_\infty$  is defined by setting

$$\mathbf{E}(x|\mathcal{A}) = \sum_k \frac{1}{m(A_k)} \left( \int_{A_k} x(s) ds \right) \chi_{A_k}.$$

Note that we do not require  $A_\infty$  to have finite measure.

Every expectation operator is a contraction both in  $L_1$  and  $L_\infty$ . Therefore,

$$\mathbf{E}(x|\mathcal{A}) \prec\prec x, \quad x \in L_1 + L_\infty.$$

It follows that  $\mathbf{E}(\cdot|\mathcal{A})$  is also contraction in  $E$ .

It will be convenient to introduce the following notation. If  $\mathcal{A}$  is a discrete subset of the semi-axis (i.e. a subset without limit points inside  $(0, \infty)$ ), then the elements of  $\mathcal{A} \cup \{0\}$  partition the semi-axis. This partition consists of a (finite or infinite) sequence of sets of finite measure. We identify this partition with the set  $\mathcal{A}$ . Elements of  $\mathcal{A}$  will be called nodes of the partition  $\mathcal{A}$ . The corresponding averaging operator will be denoted by  $\mathbf{E}(\cdot|\mathcal{A})$ .

Let  $E$  be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. Define the sets

$$\mathcal{D}_E = \text{Lin}(\{x \in E : x = \mu(x)\}) = \{\mu(a) - \mu(b), a, b \in E\},$$

$$\mathcal{Z}_E = \text{Lin}(\{x_1 - x_2 : 0 \leq x_1, x_2 \in E, \mu(x_1) = \mu(x_2)\}).$$

Let  $C$  be a Hardy operator defined by setting

$$(Cx)(t) = \frac{1}{t} \int_0^t x(s) ds.$$

The following theorem was proved in [13]. For convenience of the reader, we give a new and simple proof in the appendix.

**Theorem 8.** *Let  $E$  be a symmetric space on the semi-axis and let  $x \in \mathcal{D}_E$ . We have  $x \in Z_E$  if and only if  $Cx \in E$ . A similar assertion is also valid for the interval  $(0, 1)$  provided that  $\int_0^1 x(s)ds = 0$ .*

The following uniform submajorization was introduced by Kalton and Sukochev in [19].

Let  $x, y \in L_1(0, 1)$  (or  $x, y \in (L_1 + L_\infty)(0, \infty)$ ). We say that  $y \triangleleft x$  if there exists  $m \in \mathbb{N}$  such that

$$(6) \quad \int_{ma}^b \mu(s, y)ds \leq \int_a^b \mu(s, x)ds, \quad \forall ma \leq b.$$

Let  $x, y \in l_\infty$ . We say that  $y \triangleleft x$  if there exists  $m \in \mathbb{N}$  such that

$$(7) \quad \sum_{k=ma+1}^b \mu(k, y) \leq \sum_{k=a+1}^b \mu(k, x) \quad \forall ma+1 \leq b.$$

The following important theorem was proved in [19] (see Theorem 5.4 and Theorem 6.3 there).

**Theorem 9.** *Let  $x, y \in L_1(0, 1)$  or  $x, y \in (L_1 + L_\infty)(0, \infty)$  or  $x, y \in l_\infty$  be such that  $y \triangleleft x$ . For every  $\varepsilon > 0$ , the function  $(1 - \varepsilon)y$  belongs to a convex hull of the set  $\{z : \mu(z) \leq \mu(x)\}$ .*

This theorem led to the following fundamental result (see [19]).

**Theorem 10.** *Let  $E = E(0, 1)$  (or  $E = E(0, \infty)$  or  $E = E(\mathbb{N})$ ) be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis or on  $\mathbb{N}$ . It follows that the corresponding set  $E(\mathcal{M}, \tau)$  is a symmetric Banach space.*

Also, the uniform submajorization permits us to prove the convexity of the functional  $\pi : \mathcal{E} \rightarrow \mathbb{R}$  defined in Section 1.

**Lemma 11.** *The functional  $\pi : \mathcal{E} \rightarrow \mathbb{R}$  is convex on every symmetrically normed operator ideal  $\mathcal{E}$ .*

*Proof.* Let  $E$  be the corresponding symmetrically normed ideal of  $l_\infty$ . For every  $A, B \in \mathcal{E}$ , it follows from Proposition 8.6 of [19] that  $\mu(A + B) \triangleleft \mu(A) + \mu(B)$ . Hence,  $\sigma_m \mu(A + B) \triangleleft \sigma_m(\mu(A) + \mu(B))$ . By Theorem 9, we have

$$\|\sigma_m \mu(A + B)\|_E \leq \|\sigma_m(\mu(A) + \mu(B))\|_E \leq \|\sigma_m \mu(A)\|_E + \|\sigma_m \mu(B)\|_E.$$

Note that  $\|A^{\oplus m}\|_{\mathcal{E}} = \|\sigma_m \mu(A)\|_E$ . Dividing by  $m$  and letting  $m \rightarrow \infty$ , we obtain

$$\pi(A + B) \leq \pi(A) + \pi(B).$$

□

### 3. LIFTING OF SYMMETRIC FUNCTIONALS

In this section, we explain a canonical bijection between symmetric functionals and traces. In what follows, we require that a semifinite von Neumann algebra  $\mathcal{M}$  be either atomless or atomic with traces of all atoms being 1.

For an atomless von Neumann algebra  $\mathcal{M}$ , we have (see e.g. [12])

$$\int_0^t \mu(s, A)ds = \sup\{\tau(p|A) : p \in P(\mathcal{M}), \tau(p) = t\}, \quad A \in \mathcal{M}.$$

For a atomic von Neumann algebra  $\mathcal{M}$ , we have (see e.g. [12])

$$\sum_{k=1}^m \mu(k, A) = \sup\{\tau(p|A|) : p \in P(\mathcal{M}), \tau(p) = m\}, \quad A \in \mathcal{M}.$$

In either case, this implies a remarkable inequality (see e.g. [12])

$$(8) \quad \mu(A+B) \prec\prec \mu(A) + \mu(B) \prec\prec 2\sigma_{1/2}\mu(A+B), \quad 0 \leq A, B \in (L_1 + L_\infty)(\mathcal{M}).$$

**Lemma 12.** *Let  $E = E(0, 1)$  (or  $E = E(0, \infty)$  or  $E = E(\mathbb{N})$ ) be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis or on  $\mathbb{N}$ . If  $x, y \in E_+$  are such that  $y \triangleleft x$ , then  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional  $\varphi$  on  $E$ .*

*Proof.* Fix  $\varepsilon > 0$ . By Theorem 9, there exist  $z_k \in E$ ,  $1 \leq k \leq n$ , and positive numbers  $\lambda_k$ ,  $1 \leq k \leq n$ , such that  $\mu(z_k) \leq \mu(x)$  for every  $1 \leq k \leq n$  and

$$(1 - \varepsilon)y = \sum_{k=1}^n \lambda_k z_k, \quad \sum_{k=1}^n \lambda_k = 1.$$

Since  $\varphi$  is positive and symmetric, it follows that

$$\varphi(z_k) \leq \varphi(|z_k|) = \varphi(\mu(z_k)) \leq \varphi(\mu(x)) = \varphi(x).$$

Therefore,  $(1 - \varepsilon)\varphi(y) \leq \varphi(x)$ . Since  $\varepsilon > 0$  is arbitrarily small, the assertion follows.  $\square$

The following assertion is essentially known. However, we provide the full proof for readers convenience.

**Lemma 13.** *Let  $\mathcal{M}$  be a semifinite atomless von Neumann algebra and let  $A, B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$  be positive operators.*

$$\begin{aligned} \int_{2a}^b \mu(s, A+B) ds &\leq \int_a^b (\mu(s, A) + \mu(s, B)) ds, \quad \forall 2a \leq b, \\ \int_{2a}^b (\mu(s, A) + \mu(s, B)) ds &\leq \int_{2a}^{2b} \mu(s, A+B) ds, \quad \forall 2a \leq b. \end{aligned}$$

*Similar assertion is valid for atomic von Neumann algebra  $\mathcal{M}$ .*

*Proof.* Applying inequality (8) to the operators  $A, B$ , we obtain that

$$\int_0^b \mu(s, A+B) ds \leq \int_0^b (\mu(s, A) + \mu(s, B)) ds$$

and

$$\int_0^{2a} \mu(s, A+B) ds \geq \int_0^a (\mu(s, A) + \mu(s, B)) ds.$$

Subtracting this inequalities, we obtain

$$\int_{2a}^b \mu(s, A+B) ds \leq \int_a^b (\mu(s, A) + \mu(s, B)) ds.$$

Proof of the second inequality is identical.  $\square$

The following theorem answers Question 4 in the affirmative, as also does [20, Theorem 5.2]. The proof below is very simple and based on a completely different approach.



**Theorem 14.** *Let  $E = E(0, 1)$  (or  $E = E(0, \infty)$  or  $E = E(\mathbb{N})$ ) be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis or on  $\mathbb{N}$  and let  $E(\mathcal{M}, \tau)$  be the corresponding symmetric Banach operator space.*

- (1) *If  $\varphi$  is a positive symmetric functional on  $E$ , then there exists a positive symmetric functional  $\mathcal{L}(\varphi)$  on  $E(\mathcal{M}, \tau)$  such that  $\varphi(x) = \mathcal{L}(\varphi)(A)$  for all positive  $x \in E$  and  $A \in E(\mathcal{M}, \tau)$  such that  $\mu(A) = \mu(x)$ .*
- (2) *If  $\varphi$  is a positive symmetric functional on  $E(\mathcal{M}, \tau)$ , then there exists a positive symmetric functional  $\mathcal{L}^{-1}(\varphi)$  on  $E$  such that  $\varphi(A) = \mathcal{L}^{-1}(\varphi)(x)$  for all positive  $x \in E$  and  $A \in E(\mathcal{M}, \tau)$  such that  $\mu(A) = \mu(x)$ .*

*Proof.* We will only prove (1). Proof of (2) is identical.

Let  $A, B \in E_+(\mathcal{M}, \tau)$ . It follows from Lemma 13 that

$$\mu(A + B) \triangleleft \mu(A) + \mu(B) \triangleleft 2\sigma_{1/2}\mu(A + B).$$

It follows from Lemma 12 that

$$\varphi(\mu(A + B)) \leq \varphi(\mu(A) + \mu(B)) \leq \varphi(2\sigma_{1/2}\mu(A + B)) = \varphi(\mu(A + B)).$$

It follows that  $\mathcal{L}(\varphi)$  is additive on  $E_+(\mathcal{M}, \tau)$ . We then extend it to  $E(\mathcal{M}, \tau)$  by linearity.  $\square$

Theorem 14 provides a very natural bijection between the set of all symmetric functionals on  $E$  and that on  $E(\mathcal{M}, \tau)$ , observed first for the case of fully symmetric functionals in [8]. Next corollary follows immediately.

**Corollary 15.** *Let  $E$  and  $E(\mathcal{M}, \tau)$  be as in Theorem 14. The functional  $\varphi$  is fully symmetric on  $E$  if and only if  $\mathcal{L}(\varphi)$  is a fully symmetric functional on  $E(\mathcal{M}, \tau)$ .*

We also need a lifting between sequence and function spaces. The following space was introduced in [21].

Let  $\mathcal{A} = \{[n - 1, n]\}_{n \in \mathbb{N}}$  be a partition of the semi-axis. Clearly,  $\mathbf{E}(\cdot|\mathcal{A})$  maps  $L_1 + L_\infty$  into the set of step functions which can be identified with sequences.

**Proposition 16.** *Let  $E$  be a symmetric Banach sequence space and let  $F$  be the linear space of all such functions  $x \in L_\infty$  for which  $\mathbf{E}(\mu(x)|\mathcal{A}) \in E$ . The space  $F$  equipped with the norm*

$$\|x\|_F = \|x\|_\infty + \|\mathbf{E}(\mu(x)|\mathcal{A})\|_E$$

*is a symmetric Banach function space.*

The fact that the space  $F$  is a Banach space is non-trivial. Proof of this fact was missing in both [19] and [21]. We include it in the appendix.

Below, we assume that  $E$  is embedded into  $F$ .

**Theorem 17.** *Let  $E = E(\mathbb{N})$  be a symmetric Banach sequence space and let  $F$  be the corresponding function space.*

- (1) *If  $\varphi$  is a positive symmetric functional on  $E$ , then there exists a positive symmetric functional  $\mathcal{L}(\varphi)$  on  $F$  such that  $\varphi(\mathbf{E}(\mu(x)|\mathcal{A})) = \mathcal{L}(\varphi)(x)$  for all positive  $x \in F$ .*
- (2) *If  $\varphi$  is a positive symmetric functional on  $F$ , then its restriction on  $E$  is a positive symmetric functional. This restriction is an inverse operation for the  $\mathcal{L}$  in (1).*

*Proof.* Let us prove (1)

$$\begin{aligned}\varphi(\sigma_{1/2}a) &= 1/2\varphi(a_1, a_3, \dots) + 1/2\varphi(a_2, a_4, \dots) = \\ &= 1/2\varphi(a_1, 0, a_2, 0, \dots) + 1/2\varphi(0, a_2, 0, a_4, \dots) = 1/2\varphi(a)\end{aligned}$$

for every  $a \in E$ .

Let  $x, y \in F$  be positive. It follows from Lemma 50 that

$$\mathbf{E}(\mu(x+y)|\mathcal{A}) \triangleleft \mathbf{E}(\mu(x) + \mu(y)|\mathcal{A}) \triangleleft 2\sigma_{1/2}\mathbf{E}(\mu(x+y)|\mathcal{A}).$$

It follows from Lemma 12 that

$$\varphi(\mathbf{E}(\mu(x+y)|\mathcal{A})) = \varphi(\mathbf{E}(\mu(x) + \mu(y)|\mathcal{A}))$$

and (1) follows.

The first assertion of (2) is trivial. Clearly,  $\mu(x) - \mathbf{E}(\mu(x)|\mathcal{A}) \in (L_1 \cap L_\infty)(0, \infty)$ . If  $E \neq l_1$ , then  $\varphi(y) = 0$  for every  $y \in (L_1 \cap L_\infty)(0, \infty)$  and every symmetric functional  $\varphi$  on  $F$ . If  $E = l_1$ , then  $F = (L_1 \cap L_\infty)(0, \infty)$  and the only symmetric functional on both spaces is an integral. The second assertion of (2) follows.  $\square$

#### 4. EXISTENCE OF SYMMETRIC FUNCTIONALS

In this section, we present results concerning existence of symmetric functionals on symmetric function spaces. The main results of this section are Theorem 23, Theorem 28 and Theorem 29.

We need the following variation of the Hahn-Banach theorem.

**Lemma 18.** *Let  $E$  be a partially ordered linear space and let  $p : E \rightarrow \mathbb{R}$  be convex and monotone functional. For every  $x_0 \in E$ , there exists a positive linear functional  $\varphi : E \rightarrow \mathbb{R}$  such that  $\varphi \leq p$  and  $\varphi(x_0) = p(x_0)$ .*

*Proof.* The existence of  $\varphi$  follows from the Hahn-Banach theorem. We only have to prove that  $\varphi \geq 0$ . If  $z \geq 0$ , then  $\varphi(x_0 - z) \leq p(x_0 - z)$ . Therefore,

$$\varphi(z) \geq \varphi(x_0) - p(x_0 - z) = p(x_0) - p(x_0 - z) \geq 0$$

due to the fact that  $z \geq 0$  and  $p$  is monotone.  $\square$

Define operators  $M_m : (L_1 + L_\infty)(0, \infty) \rightarrow (L_1 + L_\infty)(0, \infty)$  (or,  $M_m : L_1(0, 1) \rightarrow L_1(0, 1)$ ) by setting

$$(M_m x)(t) = \frac{1}{t \log(m)} \int_{t/m}^t x(s) ds, \quad m \geq 2.$$

**Lemma 19.** *If  $0 \leq x \in L_1 + L_\infty$  (or,  $0 \leq x \in L_1(0, 1)$ ), then*

$$\int_a^{b/m} x(s) ds \leq \int_a^b (M_m x)(s) ds \leq \int_{a/m}^b x(s) ds$$

*provided that  $ma \leq b$ . In particular,  $m^{-1}\sigma_m x \triangleleft M_m x \triangleleft x$  provided that  $x = \mu(x)$ .*

*Proof.* Clearly,

$$\begin{aligned}\int_a^b (M_m x)(s) ds &= \frac{1}{\log(m)} \int_a^b \int_{t/m}^t x(s) ds \frac{dt}{t} = \\ &= \frac{1}{\log(m)} \int_{a/m}^b \int_{\max\{a, s\}}^{\min\{ms, b\}} \frac{dt}{t} x(s) ds = \frac{1}{\log(m)} \int_{a/m}^b x(s) \log\left(\frac{\min\{ms, b\}}{\max\{a, s\}}\right) ds.\end{aligned}$$

The integrand does not exceed  $x(s)\log(m)$  and the second inequality follows immediately. The integrand is positive and is equal to  $x(s)\log(m)$  for  $s \in (a, b/m)$ . The first inequality follows.  $\square$

**Corollary 20.** *If  $E$  is a symmetric Banach function space either on the interval  $(0, 1)$  or on the semi-axis, then  $M_m : E \rightarrow E$  is a contraction for  $m \in \mathbb{N}$ .*

*Proof.* Let  $x = \mu(x) \in E$ . It follows from Lemma 19 that  $M_m x \triangleleft x$ . It follows from theorem 9 that, for every  $\varepsilon > 0$ , the function  $(1 - \varepsilon)M_m x$  belongs to a convex hull of the set  $\{z : \mu(z) \leq \mu(x)\}$ . Therefore,  $M_m x \in E$  and  $(1 - \varepsilon)\|M_m x\|_E \leq \|x\|_E$ . Since  $\varepsilon$  is arbitrarily small, the assertion follows.  $\square$

**Lemma 21.** *Let  $E$  be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. Let  $p : \mathcal{D}_E \rightarrow \mathbb{R}$  be convex and monotone functional. If  $p = 0$  on  $Z_E \cap \mathcal{D}_E$ , then  $p$  extends to a convex monotone functional  $p : E \rightarrow \mathbb{R}$  by setting*

$$p(x) = p(\mu(x_+) - \mu(x_-)).$$

Also,  $p(x) = 0$  for every  $x \in Z_E$ .

*Proof.* If  $x \in \mathcal{D}_E$ , then  $x - \mu(x_+) + \mu(x_-) \in Z_E \cap \mathcal{D}_E$ . Therefore,  $p(x - \mu(x_+) + \mu(x_-)) = 0$  and, due to the convexity of  $p$ ,  $p(x) = p(\mu(x_+) - \mu(x_-))$ . This proves the correctness of the definition.

For  $x, y \in E$ , we have

$$\mu((x + y)_+) - \mu((x + y)_-) - \mu(x_+) + \mu(x_-) - \mu(y_+) + \mu(y_-) \in Z_E \cap \mathcal{D}_E.$$

It follows that

$$p(\mu((x + y)_+) - \mu((x + y)_-) - \mu(x_+) + \mu(x_-) - \mu(y_+) + \mu(y_-)) = 0$$

and

$$\begin{aligned} p(x + y) &= p(\mu((x + y)_+) - \mu((x + y)_-)) = \\ &= p(\mu(x_+) - \mu(x_-) + \mu(y_+) - \mu(y_-)) \leq p(x) + p(y). \end{aligned}$$

Since  $p$  is monotone on  $\mathcal{D}_E$ , then  $p(y) \leq 0$  for every  $0 \geq y \in \mathcal{D}_E$ . It follows that  $p(y) = p(-\mu(y)) \leq 0$  for  $0 \geq y \in E$ . Therefore,  $p(x + y) \leq p(x) + p(y) \leq p(x)$  for every  $0 \geq y \in E$ .  $\square$

**Lemma 22.** *Let  $E$  be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. The functional*

$$p : x \rightarrow \limsup_{m \rightarrow \infty} \|(M_m x)_+\|_E, \quad x \in \mathcal{D}_E$$

*satisfies the assumptions of Lemma 21. Also, for every  $x \in \mathcal{D}_E$ , we have  $p(x) \leq \|x\|_E$ .*

*Proof.* It follows from Corollary 20 that

$$\|(M_m x)_+\|_E \leq \|M_m x\|_E \leq \|x\|_E, \quad x \in E.$$

It follows that

$$p(x) = \limsup_{m \rightarrow \infty} \|(M_m x)_+\|_E \leq \|x\|_E, \quad x \in \mathcal{D}_E.$$

Clearly, the mappings  $x \rightarrow (M_m x)_+$  are convex and monotone. So are the mappings  $x \rightarrow \|(M_m x)_+\|_E$ . Therefore,  $p : \mathcal{D}_E \rightarrow \mathbb{R}$  is a convex and monotone functional.

If  $x \in Z_E \cap \mathcal{D}_E$ , then by Theorem 8  $|Cx| \in E$ . Therefore,

$$\begin{aligned} (M_m x)(t) &\leq \frac{1}{\log(m)} \left( \left| \frac{1}{t} \int_0^{t/m} x(s) ds \right| + \left| \frac{1}{t} \int_0^t x(s) ds \right| \right) \leq \\ &\leq \frac{1}{\log(m)} \left( \frac{1}{m} \sigma_m |Cx| + |Cx| \right)(t). \end{aligned}$$

Since  $\|\sigma_m\|_{E \rightarrow E} \leq m$  (see [22, Theorem II.4.5]), it follows that

$$\|(M_m x)_+\|_E \leq \frac{2}{\log(m)} \|Cx\|_E$$

and  $p(x) = 0$ . □

**Theorem 23.** *Let  $E = E(0, \infty)$  be a symmetric Banach space on the semi-axis. For a given  $0 \leq x \in E$ , there exists a symmetric linear functional  $\varphi : E \rightarrow \mathbb{R}$  such that*

$$\varphi(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E.$$

*Proof.* Without loss of generality,  $x = \mu(x)$ . Let  $p$  be the convex monotone functional constructed in Lemma 22. It follows from Lemma 18 that there exist a positive linear functional  $\varphi$  on  $E$  such that  $\varphi \leq p$  and  $\varphi(x) = p(x)$ . Since  $p(z) = 0$  for every  $z \in Z_E$ , it follows that  $\varphi(z) = 0$  for every  $z \in Z_E$ . Therefore,  $\varphi$  is a symmetric functional.

Since  $\varphi(z) \leq p(z) \leq \|z\|_E$  for every  $z = \mu(z) \in E$ , it follows that  $\|\varphi\|_{E^*} \leq 1$ . Therefore,

$$\varphi(x) = \varphi\left(\frac{1}{m} \sigma_m x\right) \leq \frac{1}{m} \|\sigma_m x\|_E.$$

Passing  $m \rightarrow \infty$ , we obtain

$$\varphi(x) \leq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m \mu(x)\|_E.$$

On the other hand, It follows from Lemma 19 that  $m^{-1} \sigma_m x \triangleleft M_m x$ . Therefore,

$$p(x) = \limsup_{m \rightarrow \infty} \|M_m x\|_E \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m \mu(x)\|_E.$$

The assertion follows immediately. □

Consider the functional  $\pi : E \rightarrow E$  (identical to the one defined in Section 1).

$$(9) \quad \pi(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|x^{\oplus m}\|_E, \quad x \in E.$$

Note that  $\pi(-x) = \pi(x)$  for every  $x \in E$ . If  $p$  is a functional defined in Lemma 22, then  $p(-x) = 0$  for positive  $x \in E$ . Therefore,  $p \neq \pi$ . However, the assertion below follows from Theorem 23.

**Lemma 24.** *Let  $E = E(0, \infty)$  be a symmetric Banach space on the semi-axis. Let  $p$  and  $\pi$  be the convex functionals on  $E$  defined in Lemma 22 and (9), respectively. For every positive  $x \in E$ , we have  $p(x) = \pi(x)$ .*

*Proof.* For every  $x \in E$ , consider the functional  $\varphi$  constructed in Theorem 23. By construction, we have  $\varphi(x) = p(x) = \pi(x)$ . □

If  $E \not\subset L_1(0, \infty)$ , then the functional  $\varphi$  constructed in Theorem 23 is necessarily singular. The case  $E \subset L_1$  requires more detailed treatment.

**Lemma 25.** *Let  $E$  be a symmetric (respectively, fully symmetric) Banach function space either on the interval  $(0, 1)$  or on the semi-axis. Let  $\{\varphi_i\}_{i \in \mathbb{I}} \in E^*$  be a net and let  $\varphi \in E^*$  be such that  $\varphi_i \rightarrow \varphi$   $*$ -weakly.*

- (1) *If every  $\varphi_i$  is symmetric, then  $\varphi$  is symmetric.*
- (2) *If every  $\varphi_i$  is fully symmetric, then  $\varphi$  is fully symmetric.*

*Proof.* Let each  $\varphi_i$  be symmetric. If  $0 \leq x_1, x_2 \in E$  are such that  $\mu(x_1) = \mu(x_2)$ , then

$$\varphi(x_1) = \lim_{i \in \mathbb{I}} \varphi_i(x_1) = \lim_{i \in \mathbb{I}} \varphi_i(x_2) = \varphi(x_2).$$

Hence,  $\varphi$  is symmetric.

Let each  $\varphi_i$  be fully symmetric. Thus,  $\varphi_i(x) \leq 0$  for every  $x \in \mathcal{D}_E$  such that  $Cx \leq 0$ . Therefore,  $\varphi(x) = \lim_{i \in \mathbb{I}} \varphi_i(x) \leq 0$  for every  $x \in \mathcal{D}_E$  such that  $Cx \leq 0$ .

Let  $x_1, x_2 \in E$  be positive elements such that  $x_1 \prec\prec x_2$ . Therefore,  $z = \mu(x_1) - \mu(x_2) \in \mathcal{D}_E$  and  $Cz \leq 0$ . It follows from above that  $\varphi(z) \leq 0$ . Hence,  $\varphi$  is a fully symmetric functional.  $\square$

**Lemma 26.** *Let  $E$  be a symmetric (respectively, fully symmetric) Banach function space either on the interval  $(0, 1)$  or on the semi-axis and let  $\varphi$  be a symmetric (respectively, fully symmetric) functional on  $E$ . The formula*

$$\varphi_{\text{sing}}(x) = \lim_{n \rightarrow \infty} \varphi(\mu(x)\chi_{(0, 1/n)}), \quad 0 \leq x \in E.$$

*defines a singular symmetric (respectively, fully symmetric) linear functional on  $E$ .*

*Proof.* If  $x, y \in E$  are positive functions, then

$$\mu(x + y)\chi_{(0, 1/n)} \triangleleft (\mu(x) + \mu(y))\chi_{(0, 1/n)} \triangleleft 2\sigma_{1/2}\mu(x + y)\chi_{(0, 1/n)}.$$

Taking the limit as  $n \rightarrow \infty$ , we derive from Lemma 12 that

$$\varphi_{\text{sing}}(\mu(x + y)) = \varphi_{\text{sing}}(\mu(x) + \mu(y)).$$

Since  $\varphi$  is symmetric, it follows that

$$\varphi_{\text{sing}}(x + y) = \varphi_{\text{sing}}(\mu(x + y)) = \varphi_{\text{sing}}(\mu(x) + \mu(y)) = \varphi_{\text{sing}}(x) + \varphi_{\text{sing}}(y).$$

Hence,  $\varphi_{\text{sing}}$  is an additive functional on  $E_+$ . Therefore, it extends to a linear functional on  $E$ . Clearly,  $\varphi_{\text{sing}}$  is symmetric. Second assertion is trivial.  $\square$

In fact, the construction in Lemma 26 gives a singular part of the functional  $\varphi$  as defined by Yosida-Hewitt theorem.

**Lemma 27.** *Let  $E = E(0, \infty) \subset L_1(0, \infty)$  be a symmetric Banach function space on the semi-axis and let  $\varphi$  be a symmetric functional on  $E$ . If  $\varphi_{\text{sing}}$  is a functional constructed in Lemma 26, then  $\varphi - \varphi_{\text{sing}}$  is a normal functional (that is, an integral).*

*Proof.* It is clear that

$$0 \leq \varphi_{\text{sing}}(z) \leq \|z\|_\infty \lim_{n \rightarrow \infty} \varphi(\chi_{(0, 1/n)}) = 0$$

for every positive  $z \in (L_1 \cap L_\infty)(0, \infty)$ . It follows that  $\varphi_{\text{sing}}(\mu(x)\chi_{(1/n, \infty)}) = 0$  for every  $x \in E$ . Therefore,

$$(10) \quad (\varphi - \varphi_{\text{sing}})(x) = \lim_{n \rightarrow \infty} \varphi(\mu(x)\chi_{(1/n, \infty)}) = \lim_{n \rightarrow \infty} (\varphi - \varphi_{\text{sing}})(\mu(x)\chi_{(1/n, \infty)}).$$

On the other hand, for every positive  $z \in (L_1 \cap L_\infty)(0, \infty)$  with  $\|z\|_\infty = 1$ , we have  $z \prec \chi_{(0, \|z\|_1)}$ . It is proved in [30, Theorem 23] that  $z$  belongs to the closure (in the topology of  $L_1 \cap L_\infty$ ) of the set  $\{u \geq 0 : \mu(u) = \chi_{(0, \|z\|_1)}\}$ . Thus,

$$(\varphi - \varphi_{sing})(z) = (\varphi - \varphi_{sing})(\chi_{(0, \|z\|_1)}) = \|z\|_1(\varphi - \varphi_{sing})(\chi_{(0,1)}).$$

By linearity,

$$(11) \quad (\varphi - \varphi_{sing})(z) = (\varphi - \varphi_{sing})(\chi_{(0,1)}) \cdot \int_0^\infty z(s)ds, \quad \forall z \in (L_1 \cap L_\infty)(0, \infty).$$

It follows that from (10) and (11) that

$$(\varphi - \varphi_{sing})(x) = \lim_{n \rightarrow \infty} \int_{1/n}^\infty \mu(s, x)ds \cdot (\varphi - \varphi_{sing})(0, 1) = \int_0^1 x(s)ds \cdot (\varphi - \varphi_{sing})(\chi_{(0,1)})$$

for every positive function  $x \in E$ . The assertion follows immediately.  $\square$

**Theorem 28.** *Let  $E \subset L_1(0, \infty)$  be a symmetric Banach space on the semi-axis. For a given  $0 \leq x \in E$ , there exists a singular symmetric linear functional  $\varphi_{sing}$  such that*

$$\varphi_{sing}(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E.$$

*Proof.* Apply Theorem 23 to the function  $\mu(x)\chi_{(0,1/n)}$ . It follows that there exists a symmetric linear functional  $\varphi_n$  such that  $\|\varphi_n\|_{E^*} \leq 1$  and

$$\varphi_n(\mu(x)\chi_{(0,1/n)}) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x)\chi_{(0,1/n)})\|_E \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E.$$

Since the unit ball in  $E^*$  is  $*$ -weakly compact (Banach-Alaoglu theorem), there exists a convergent subnet  $\psi_i = \varphi_{F(i)}$ ,  $i \in \mathbb{I}$ , of the sequence  $\varphi_n$ ,  $n \in \mathbb{N}$ . Let  $\psi_i \rightarrow \varphi$ . It follows from Lemma 25 that  $\varphi$  is a symmetric functional.

By the definition of a subnet (see [26, Section IV.2]), for every fixed  $n \in \mathbb{N}$ , there exists  $i_n \in \mathbb{I}$  such that  $F(i) > n$  for every  $i > i_n$ . Thus, for every  $i > i_n$ , we have

$$\psi_i(\mu(x)\chi_{(0,1/n)}) \geq \varphi_{F(i)}(\mu(x)\chi_{(0,1/F(i))}) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E.$$

The subnet  $\psi_i$ ,  $i_n < i \in \mathbb{I}$  converges to the same limit  $\varphi$ . Therefore,

$$\varphi(\mu(x)\chi_{(0,1/n)}) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E.$$

Now, taking the limit as  $n \rightarrow \infty$ , we obtain the inequality

$$\varphi_{sing}(x) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E,$$

where  $\varphi_{sing}$  is a singular symmetric functional defined in Lemma 26. The opposite inequality is trivial.  $\square$

**Theorem 29.** *Let  $E$  be a symmetric Banach space on the interval  $(0, 1)$ . For a given  $0 \leq x \in E$ , there exists a singular symmetric linear functional  $\varphi_{sing}$  such that*

$$\varphi_{sing}(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E.$$

*Proof.* Let  $F$  be a symmetric Banach space on the semi-axis with a norm given by the formula

$$\|x\|_F = \|\mu(x)\chi_{(0,1)}\|_E + \|x\|_1, \quad \forall x \in F.$$

Clearly,  $F \subset L_1(0, \infty)$ . Applying Theorem 28, we obtain a symmetric singular functional  $\varphi$  on  $F$  such that

$$\varphi(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_F = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E.$$

□

## 5. EXISTENCE OF FULLY SYMMETRIC FUNCTIONALS

In this section, we present results concerning existence of fully symmetric functionals on fully symmetric function spaces. The main results of this section are Theorem 33, Theorem 35 and Theorem 36.

**Lemma 30.** *Let  $E$  be a symmetric Banach function space either on the interval  $(0, 1)$  or on the semi-axis. If  $x, z \in \mathcal{D}_E$  are such that  $Cx \leq Cz$ , then  $CM_m x \leq CM_m z$ .*

*Proof.* Let  $x = \mu(a) - \mu(b)$  and  $z = \mu(c) - \mu(d)$  with  $a, b, c, d \in E$ . It follows from assumption  $Cx \leq Cz$  that  $C(\mu(a) + \mu(d)) \leq C(\mu(b) + \mu(c))$  or, equivalently,  $\mu(a) + \mu(d) \prec\prec \mu(b) + \mu(c)$ .

Arguing as in Lemma 19, we have

$$\int_0^t (M_m z)(s) ds = \int_0^t z(s) h(s, t) ds$$

with

$$h(s, t) = \begin{cases} 1, & 0 \leq s \leq t/m \\ \frac{\log(t/s)}{\log(m)}, & t/m \leq s \leq t \end{cases}$$

It is now clear that

$$\begin{aligned} \int_0^t M_m(\mu(a) + \mu(d))(s) ds &= \int_0^t (\mu(s, a) + \mu(s, d)) h(s, t) ds, \\ \int_0^t M_m(\mu(b) + \mu(c))(s) ds &= \int_0^t (\mu(s, b) + \mu(s, c)) h(s, t) ds. \end{aligned}$$

Clearly,  $h$  is positive and decreasing with respect to  $s$ . It follows from [22, Equality 2.36] that

$$M_m(\mu(a) + \mu(d)) \prec\prec M_m(\mu(b) + \mu(c))$$

and the assertion follows. □

**Lemma 31.** *Let  $E$  be a fully symmetric Banach function space either on the interval  $(0, 1)$  or on the semi-axis and let  $x = \mu(x) \in E$ . If  $z \in \mathcal{D}_E$  is such that  $Cx \leq Cz$ , then  $p(x) \leq p(z)$ .*

*Proof.* Since  $M_m x$  is decreasing, it follows from Lemma 30 that

$$\int_0^t \mu(s, M_m x) ds = \int_0^t (M_m x)(s) ds \leq \int_0^t (M_m z)_+(s) ds \leq \int_0^t \mu(s, (M_m z)_+) ds.$$

Therefore,  $(M_m x)_+ = M_m x \prec\prec (M_m z)_+$ . The assertion follows now from the definition of the functional  $p$ . □

**Lemma 32.** *Let  $E$  be a fully symmetric Banach function space either on the interval  $(0, 1)$  or on the semi-axis. Let  $p$  be the functional constructed in Lemma 22. The functional*

$$q(x) = \inf\{p(z) : z \in \mathcal{D}_E, Cx \leq Cz\}, \quad x \in \mathcal{D}_E$$

*satisfies the assumptions of Lemma 21.*

*Proof.* It is clear from the definition of  $q$  that  $q \leq p$  and that  $q$  is a positive functional.

We claim that  $q$  is convex on  $\mathcal{D}_E$ . Let  $x_1, x_2 \in \mathcal{D}_E$ . Fix  $\varepsilon > 0$  and select  $z_1, z_2 \in \mathcal{D}_E$  such that  $Cx_i \leq Cz_i$  and  $p(z_i) \leq q(x_i) + \varepsilon$  for  $i = 1, 2$ . Thus,  $C(x_1 + x_2) \leq C(z_1 + z_2)$  and

$$q(x_1 + x_2) \leq p(z_1 + z_2) \leq p(z_1) + p(z_2) \leq q(x_1) + q(x_2) + 2\varepsilon.$$

Since  $\varepsilon$  is arbitrarily small, the claim follows.

We claim that  $q$  is monotone on  $\mathcal{D}_E$ . Let  $x_1, x_2 \in \mathcal{D}_E$  be such that  $x_1 \leq x_2$ . Fix  $\varepsilon > 0$  and select  $z \in \mathcal{D}_E$  such that  $Cx_2 \leq Cz$  and  $p(z) \leq q(x_2) + \varepsilon$ . Thus,  $Cx_1 \leq Cx_2 \leq Cz$  and  $q(x_1) \leq p(z) \leq q(x_2) + \varepsilon$ . Since  $\varepsilon$  is arbitrarily small, the claim follows.

For  $x \in Z_E \cap \mathcal{D}_E$ , we have  $0 \leq q(x) \leq p(x) = 0$  and, therefore,  $q(x) = 0$ .  $\square$

The following theorem is the first main result of this section.

**Theorem 33.** *Let  $E = E(0, \infty)$  be a fully symmetric Banach space on the semi-axis. For a given  $0 \leq x \in E$ , there exists a fully symmetric linear functional  $\varphi : E \rightarrow \mathbb{R}$  such that*

$$\varphi(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E.$$

*Proof.* Without loss of generality,  $x = \mu(x)$ . Let  $q$  be the convex monotone functional constructed in Lemma 32. It follows from Lemma 18 that there exist a positive linear functional  $\varphi$  on  $E$  such that  $\varphi \leq q$  and  $\varphi(x) = q(x)$ .

It is clear that  $\varphi \leq q \leq p$ . Since  $p(z) = 0$  for every  $z \in Z_E$ , it follows that  $\varphi(z) = 0$  for every  $z \in Z_E$ . Therefore,  $\varphi$  is a symmetric functional. For every  $z \in \mathcal{D}_E$  with  $Cz \leq 0$ , we have  $\varphi(z) \leq q(z) \leq p(0) = 0$ .

Let  $x_1, x_2 \in E$  be positive elements such that  $x_1 \prec\prec x_2$ . Therefore,  $z = \mu(x_1) - \mu(x_2) \in \mathcal{D}_E$  and  $Cz \leq 0$ . It follows from above that  $\varphi(z) \leq 0$ . Hence,  $\varphi$  is a fully symmetric functional.

Since  $\varphi(z) \leq q(z) \leq p(z) \leq \|z\|_E$  for every  $z = \mu(z) \in E$ , it follows that  $\|\varphi\|_{E^*} \leq 1$ . Therefore,

$$\varphi(x) = \varphi\left(\frac{1}{m}\sigma_m x\right) \leq \frac{1}{m} \|\sigma_m x\|_E.$$

Passing  $m \rightarrow \infty$ , we obtain

$$\varphi(x) \leq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m \mu(x)\|_E.$$

On the other hand,  $q(x) = p(x)$  by Lemma 31. By Lemma 19, we have  $m^{-1}\sigma_m x \prec M_m x$ . Therefore,

$$\varphi(x) = q(x) = p(x) = \limsup_{m \rightarrow \infty} \|M_m x\|_E \geq \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m \mu(x)\|_E.$$

The assertion follows immediately.  $\square$



If  $\pi : E \rightarrow E$  is a convex functional defined in (9), then  $\pi(-x) = \pi(x)$  for every  $x \in E$ . If  $q$  is a functional defined in Lemma 32, then  $q(-x) = 0$  for positive  $x \in E$ . Therefore,  $q \neq \pi$ . However, the assertion below follows from Theorem 33.

**Lemma 34.** *Let  $E = E(0, \infty)$  be a fully symmetric Banach space on the semi-axis. Let  $q$  and  $\pi$  be the convex functionals on  $E$  defined in Lemma 32 and (9), respectively. For every positive  $x \in E$ , we have  $q(x) = \pi(x)$ .*

*Proof.* For every  $x \in E$ , consider the functional  $\varphi$  constructed in Theorem 33. By construction, we have  $\varphi(x) = q(x) = p(x) = \pi(x)$ .  $\square$

The proofs of the two following theorems are very similar to that of Theorem 28 (respectively, Theorem 29) and are, therefore, omitted. The only difference is that the reference to Theorem 23 (respectively, Theorem 28) has to be replaced with the reference to Theorem 33 (respectively, Theorem 35).

**Theorem 35.** *Let  $E \subset L_1(0, \infty)$  be a fully symmetric Banach space on the semi-axis. For a given  $0 \leq x \in E$ , there exists a singular fully symmetric linear functional  $\varphi_{sing}$  such that*

$$\varphi_{sing}(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E.$$

**Theorem 36.** *Let  $E \subset L_1(0, 1)$  be a fully symmetric Banach space on the interval  $(0, 1)$ . For a given  $0 \leq x \in E$ , there exists a singular fully symmetric linear functional  $\varphi_{sing}$  such that*

$$\varphi_{sing}(x) = \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E.$$

## 6. THE SETS OF SYMMETRIC AND FULLY SYMMETRIC FUNCTIONALS ARE DIFFERENT

In this section, we demonstrate that the sets of symmetric and fully symmetric functionals on a given fully symmetric space  $E$  are distinct (provided that one of these sets is non-empty). The main results are Theorem 47 and Theorem 48.

Let  $x = \mu(x) \in (L_1 + L_\infty)(0, \infty)$  (or  $x = \mu(x) \in L_1(0, 1)$ ) and let  $X(t) = \int_0^t x(s)ds$ . For every  $\theta > 0$ , let  $a_n(\theta)$  be such that

$$X(a_n(\theta)) = (3/2)^n \theta$$

for every  $n \in \mathbb{Z}$  such that  $a_n(\theta)$  does exist. Given a sequence  $\kappa = \{\kappa_n\}_{n \in \mathbb{Z}} \in (\mathbb{N} \cup \{\infty\})^{\mathbb{Z}}$ , let

$$\mathcal{B}_{\kappa, \theta} = \{\kappa_n a_{3n}(\theta), \text{ where } n \in \mathbb{Z} \text{ is such that } \kappa_n^2 a_{3n}(\theta) < a_{3n+1}(\theta)\}.$$

If  $\kappa_n = m$  for all  $n \in \mathbb{N}$ , we write  $\mathcal{B}_{m, \theta}$  instead of  $\mathcal{B}_{\kappa, \theta}$ . Also, set

$$\mathcal{A}_m = \{ma_n(1) : m^2 a_n(1) < a_{n+1}(1), n \in \mathbb{Z}\}.$$

**Lemma 37.** *If  $x = \mu(x) \in L_1 + L_\infty$  and if  $\mathcal{C}_i$ ,  $1 \leq i \leq k$ , are discrete sets, then*

$$\mathbf{E}(x | \cup_{i=1}^k \mathcal{C}_i) \prec \sum_{i=1}^k \mathbf{E}(x | \mathcal{C}_i).$$

*Proof.* It is sufficient to verify

$$\int_0^t \mathbf{E}(x | \cup_{i=1}^k C_i)(s) ds \leq \sum_{i=1}^k \int_0^t \mathbf{E}(x | C_i)(s) ds$$

only at the nodes of  $\mathbf{E}(x | \cup_{i=1}^k C_i)$ , that is at the nodes of  $\mathbf{E}(x | C_i)$  for every  $i$ . However, if  $t \in C_i$  for some  $i$ , then

$$\int_0^t \mathbf{E}(x | \cup_{i=1}^k C_i)(s) ds = X(t) = \int_0^t \mathbf{E}(x | C_i)(s) ds$$

and we are done.  $\square$

We will need the following lemma.

**Lemma 38.** *If  $x = \mu(x) \in L_1 + L_\infty$  and if  $\kappa \geq \kappa'$  (that is  $\kappa_n \geq \kappa'_n$  for every  $n$ ), then*

$$(12) \quad \mathbf{E}(x | \mathcal{B}_{\kappa, \theta}) \prec \prec \frac{3}{2} \mathbf{E}(x | \mathcal{B}_{\kappa', \theta}).$$

*Proof.* Let  $n \in \mathbb{Z}$  be such that  $\kappa_n^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ . It follows that  $\kappa_n'^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ . Therefore,

$$\begin{aligned} \int_0^{\kappa_n a_{3n}(\theta)} \mathbf{E}(x | \mathcal{B}_{\kappa, \theta})(s) ds &\leq \int_0^{a_{3n+1}(\theta)} x(s) ds = 3/2 \int_0^{a_{3n}(\theta)} x(s) ds \leq \\ &\leq 3/2 \int_0^{\kappa_n' a_{3n}(\theta)} x(s) ds = 3/2 \int_0^{\kappa_n' a_{3n}(\theta)} \mathbf{E}(x | \mathcal{B}_{\kappa', \theta})(s) ds. \end{aligned}$$

Hence, we have

$$(13) \quad \int_0^t \mathbf{E}(x | \mathcal{B}_{\kappa, \theta})(s) ds \leq 3/2 \int_0^t \mathbf{E}(x | \mathcal{B}_{\kappa', \theta})(s) ds$$

for every  $t$  being a node of the partition  $\mathcal{B}_{\kappa, \theta}$ . Thus, (13) holds for every  $t > 0$ .  $\square$

**Remark 39.** *The inequality (12) holds if  $\kappa_n \geq \kappa'_n$  only for such  $n \in \mathbb{Z}$  that satisfy the inequality  $\kappa_n^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ .*

**Lemma 40.** *Let  $E$  be a fully symmetric Banach function space either on the interval  $(0, 1)$  or on the semi-axis. Let  $x = \mu(x) \in E$  and  $y = \mu(y) \in E$  be such that  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional  $\varphi \in E^*$ . There exists  $0 \leq u_m \in E$  such that  $u_m \rightarrow 0$  in  $E$  and*

$$\int_{ma}^b y(s) ds \leq \int_a^{mb} (x + u_m)(s) ds, \quad \forall ma \leq b.$$

*Proof.* Let  $p$  be a convex positive functional considered in Lemma 22. By Lemma 18, there exists a positive functional  $\varphi \in E^*$  such that  $\varphi \leq p$  and  $\varphi(y - x) = p(y - x)$ . We have  $p(z) = 0$  for every  $z \in Z_E$  and, therefore,  $\varphi(z) = 0$  for every  $z \in Z_E$ . Therefore,  $\varphi$  is a positive symmetric linear functional on  $E$ .

By the assumption,  $\varphi(y - x) \leq 0$  and, therefore,  $p(y - x) = 0$ . Hence, by the definition of  $p$ , we have  $u_m = (M_m(y - x))_+ \rightarrow 0$  in  $E$ . Clearly,  $M_m y \leq M_m x + u_m$ . It follows from Lemma 19 that

$$\int_{ma}^b y(s) ds \leq \int_{ma}^{mb} (M_m y)(s) ds \leq \int_{ma}^{mb} (M_m x + u_m)(s) ds \leq \int_a^{mb} (x + u_m)(s) ds.$$

$\square$

For each sequence  $\kappa$  and  $\lambda > 0$ , we define the sequence  $\kappa^\lambda$  by setting

$$\kappa_n^\lambda = \begin{cases} \kappa_n, & \kappa_n \geq \lambda \\ \infty, & \kappa_n < \lambda. \end{cases}$$

**Lemma 41.** *If  $m \in \mathbb{N}$ ,  $x = \mu(x) \in L_1 + L_\infty$  and  $0 \leq u \in L_1 + L_\infty$  are such that*

$$\int_{ma}^b \mathbf{E}(x|\mathcal{B}_{\kappa,\theta})(s)ds \leq \int_a^{mb} (x+u)(s)ds, \quad \forall ma \leq b \in \mathbb{R},$$

then

$$(14) \quad m^{-1}\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta}) \prec\prec 30\mu(u).$$

*Proof.* If  $\kappa_n^{100m} = \infty$  for every  $n \in \mathbb{Z}$ , then  $\mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta}) = 0$  and the assertion is trivial.

Let  $n \in \mathbb{Z}$  be such that  $\kappa_n^2 a_{3n}(\theta) < a_{3n+1}(\theta)$  and  $\kappa_n \geq 100m$ . It follows that

$$(15) \quad \int_0^{m\kappa_n a_{3n}(\theta)} u(s)ds \geq \int_{a_{3n}(\theta)}^{m\kappa_n a_{3n}(\theta)} (x+u)(s)ds - \int_{a_{3n}(\theta)}^{m\kappa_n a_{3n}(\theta)} x(s)ds.$$

By the assumption, we have

$$(16) \quad \int_{a_{3n}(\theta)}^{m\kappa_n a_{3n}(\theta)} (x+u)(s)ds \geq \int_{ma_{3n}(\theta)}^{\kappa_n a_{3n}(\theta)} \mathbf{E}(x|\mathcal{B}_{\kappa,\theta})(s)ds.$$

Note that  $m\kappa_n a_{3n}(\theta) < a_{3n+1}(\theta)$ . It follows from (15) and (16) that

$$(17) \quad \int_0^{m\kappa_n a_{3n}(\theta)} u(s)ds \geq \int_{ma_{3n}(\theta)}^{\kappa_n a_{3n}(\theta)} \mathbf{E}(x|\mathcal{B}_{\kappa,\theta})(s)ds - \int_{a_{3n}(\theta)}^{a_{3n+1}(\theta)} x(s)ds.$$

Let  $n'$  be the maximal integer number such that  $n' < n$  and  $\kappa_{n'}^2 a_{3n'}(\theta) < a_{3n'+1}(\theta)$ . It is clear that

$$\kappa_{n'}^2 a_{3n'}(\theta) < a_{3n'+1}(\theta) \leq a_{3n-2}(\theta) < ma_{3n}(\theta)$$

and

$$(18) \quad \mathbf{E}(x|\mathcal{B}_{\kappa,\theta}) = \frac{X(\kappa_n a_{3n}(\theta)) - X(\kappa_{n'} a_{3n'}(\theta))}{\kappa_n a_{3n}(\theta) - \kappa_{n'} a_{3n'}(\theta)} \geq \frac{X(a_{3n}(\theta)) - X(a_{3n-2}(\theta))}{\kappa_n a_{3n}(\theta)}$$

on the interval  $(ma_{3n}(\theta), \kappa_n a_{3n}(\theta))$ .

If  $\kappa_{n'}^2 a_{3n'}(\theta) \geq a_{3n+1}(\theta)$  for every  $n' < n$ , then

$$(19) \quad \mathbf{E}(x|\mathcal{B}_{\kappa,\theta}) = \frac{X(\kappa_n a_{3n}(\theta))}{\kappa_n a_{3n}(\theta)} \geq \frac{X(a_{3n}(\theta))}{\kappa_n a_{3n}(\theta)}$$

on the interval  $(ma_{3n}(\theta), \kappa_n a_{3n}(\theta))$ .

It follows from (17) and (18) (or (19)) that

$$\int_0^{m\kappa_n a_{3n}(\theta)} u(s)ds \geq \frac{\kappa_n - m}{\kappa_n} \cdot \left(1 - \frac{4}{9}\right) X(a_{3n}(\theta)) - \frac{1}{2} X(a_{3n}(\theta)).$$

Since  $\kappa_n \geq 100m$ , it follows that

$$\begin{aligned} \int_0^{m\kappa_n a_{3n}(\theta)} u(s)ds &\geq \left(\left(1 - \frac{1}{100}\right)\left(1 - \frac{4}{9}\right) - \frac{1}{2}\right) X(a_{3n}(\theta)) = \frac{1}{20} X(a_{3n}(\theta)) = \\ &= \frac{1}{30} X(a_{3n+1}(\theta)) \geq \frac{1}{30} X(\kappa_n a_{3n}(\theta)) = \frac{1}{30} \int_0^{\kappa_n a_{3n}(\theta)} \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta})(s)ds. \end{aligned}$$

It follows immediately that

$$(20) \quad \int_0^t \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta})(s)ds \leq 30 \int_0^{mt} u(s)ds \leq 30 \int_0^{mt} \mu(s,u)ds$$

for every  $t$  being a node of the partition  $\mathcal{B}_{\kappa^{100m},\theta}$ . Therefore,

$$\int_0^t \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta})(s)ds \leq 30 \int_0^{mt} \mu(s,u)ds, \quad t > 0$$

or, equivalently,

$$\int_0^{t/m} \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta})(s)ds \leq 30 \int_0^t \mu(s,u)ds, \quad t > 0.$$

The assertion follows immediately.  $\square$

**Lemma 42.** *Let  $E$  be a fully symmetric Banach function space either on the interval  $(0,1)$  or on the semi-axis. If  $x = \mu(x) \in E$  is such that  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional  $\varphi$  on  $E$  and every  $0 \leq y \prec\prec x$ , then  $\lambda^{-1}\sigma_\lambda \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta}) \rightarrow 0$  as  $\lambda \rightarrow \infty$ .*

*Proof.* Since  $\mathbf{E}(x|\mathcal{B}_{\kappa,\theta}) \prec\prec x$ , it follows from the assumption and Lemma 40 that there exists  $0 \leq u_m \rightarrow 0$  such that

$$\int_{ma}^b \mathbf{E}(x|\mathcal{B}_{\kappa,\theta})(s)ds \leq \int_a^{mb} (x + u_m)(s)ds, \quad \forall ma \leq b \in \mathbb{R}.$$

For every  $\lambda \geq 100m$ , we have  $\kappa^{100m} \leq \kappa^\lambda$ . It follows from Lemma 41 that

$$\frac{1}{\lambda}\sigma_\lambda \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta}) \prec\prec \frac{1}{m}\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta}) \stackrel{\text{Lemma 38}}{\prec\prec} \frac{3}{2m}\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{100m},\theta}) \stackrel{(14)}{\prec\prec} 45\mu(u_m).$$

The assertion now follows immediately.  $\square$

**Proposition 43.** *Let  $E$  be a fully symmetric Banach function space either on the interval  $(0,1)$  or on the semi-axis equipped with a Fatou norm. If  $x = \mu(x) \in E$  is such that  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional  $\varphi$  on  $E$  and every  $0 \leq y \prec\prec x$ , then  $m^{-1}\sigma_m \mathbf{E}(x|\mathcal{B}_{m,\theta}) \rightarrow 0$  as  $m \rightarrow \infty$ .*

*Proof.* For every  $m, r \in \mathbb{N}$ , set

$$\kappa_n^{m,r} = \begin{cases} m & 0 \leq |n| < r \\ \infty & r \leq |n| \end{cases}$$

and  $\kappa^{m,r} = \{\kappa_n^{m,r}\}_{n \in \mathbb{Z}}$ . Clearly,  $\mathbf{E}(x|\mathcal{B}_{\kappa^{m,r},\theta}) \rightarrow \mathbf{E}(x|\mathcal{B}_{m,\theta})$  almost everywhere when  $r \rightarrow \infty$ . It follows from the definition of Fatou norm that

$$\lim_{r \rightarrow \infty} \|\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{m,r},\theta})\|_E = \|\sigma_m \mathbf{E}(x|\mathcal{B}_{m,\theta})\|_E.$$

Select  $r_m$  so large that

$$(21) \quad \frac{1}{m} \|\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{m,r_m},\theta})\|_E > \frac{1}{2m} \|\sigma_m \mathbf{E}(x|\mathcal{B}_{m,\theta})\|_E.$$

Now define the sequence  $\kappa = \{\kappa_n\}_{n \in \mathbb{Z}}$  by setting

$$\kappa_n = \inf_{m \geq 1} \kappa_n^{m,r_m} = \inf_{r_m > |n|} m, \quad n \in \mathbb{Z}.$$

Clearly,  $r_{\kappa_n} \geq |n|$  and, therefore,  $\kappa_n \rightarrow \infty$  as  $|n| \rightarrow \infty$ . In particular, the set  $\{n : \kappa_n < \lambda\}$  is finite for every  $\lambda \in \mathbb{N}$ . Set

$$M(\lambda) = \max\{\lambda, \max_{\kappa_n < \lambda} \left(\frac{a_{3n+1}(\theta)}{a_{3n}(\theta)}\right)^{1/2}\}.$$

If  $m > M(\lambda)$ , then  $m^2 a_{3n}(\theta) \geq a_{3n+1}(\theta)$  whenever  $\kappa_n < \lambda$ . Thus,  $\kappa_n \geq \lambda$  whenever  $m^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ . Hence,  $\kappa_n^\lambda = \kappa_n$  whenever  $(\kappa_n^{m,r_m})^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ . Therefore,  $\kappa_n^\lambda \leq \kappa_n^{m,r_m}$  for every  $n \in \mathbb{Z}$  such that  $(\kappa_n^{m,r_m})^2 a_{3n}(\theta) < a_{3n+1}(\theta)$ . According to Remark 39, it follows that

$$\mathbf{E}(x|\mathcal{B}_{\kappa^{m,r_m},\theta}) \prec\prec \frac{3}{2}\mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta}).$$

Since  $m \geq \lambda$ , it follows that

$$(22) \quad \frac{1}{m}\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{m,r_m},\theta}) \prec\prec \frac{3}{2\lambda}\sigma_\lambda \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta}).$$

By Lemma 42, for every  $\varepsilon > 0$ , there exists  $\lambda$  such that

$$(23) \quad \frac{1}{\lambda}\|\sigma_\lambda \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta})\|_E < \frac{1}{3}\varepsilon.$$

It follows that

$$\frac{1}{m}\|\sigma_m \mathbf{E}(x|\mathcal{B}_{m,\theta})\|_E \stackrel{(21)}{\leq} \frac{2}{m}\|\sigma_m \mathbf{E}(x|\mathcal{B}_{\kappa^{m,r_m},\theta})\|_E \stackrel{(22)}{\leq} \frac{3}{\lambda}\|\sigma_\lambda \mathbf{E}(x|\mathcal{B}_{\kappa^\lambda,\theta})\|_E \stackrel{(23)}{<} \varepsilon$$

for every  $m > M(\lambda)$ . Since  $\varepsilon > 0$  is arbitrarily small, the assertion follows.  $\square$

**Lemma 44.** *Let  $E$  be a fully symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis equipped with a Fatou norm. If  $x = \mu(x) \in E$  is such that  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional  $\varphi$  on  $E$  and every  $0 \leq y \prec\prec x$ , then  $m^{-1}\sigma_m \mathbf{E}(x|\mathcal{A}_m) \rightarrow 0$  as  $m \rightarrow \infty$ .*

*Proof.* It is clear that  $a_k(3/2) = a_{k+1}(1)$  and  $a_k((3/2)^2) = a_{k+2}(1)$  for every  $k \in \mathbb{N}$ . It follows that

$$\mathcal{B}_{m,1} \cup \mathcal{B}_{m,3/2} \cup \mathcal{B}_{m,(3/2)^2} = \mathcal{A}_m.$$

Therefore, by Lemma 37, we have

$$(24) \quad \mathbf{E}(x|\mathcal{A}_m) \prec\prec \mathbf{E}(x|\mathcal{B}_{m,1}) + \mathbf{E}(x|\mathcal{B}_{m,3/2}) + \mathbf{E}(x|\mathcal{B}_{m,(3/2)^2}).$$

The assertion follows now from Proposition 43.  $\square$

**Lemma 45.** *Let  $x = \mu(x) \in L_1 + L_\infty(0, \infty)$  be a function on the semi-axis. If  $x \notin L_1(0, \infty)$ , then, for every  $t > 0$  and every  $m \in \mathbb{N}$ , we have*

$$(25) \quad X(t) \leq \frac{2}{3}X(m^4 t) + \frac{3}{2} \int_0^{m^4 t} \mathbf{E}(x|\mathcal{A}_m)(s) ds.$$

*Proof.* For a given  $t > 0$ , there exists  $n \in \mathbb{Z}$  such that  $t \in [a_n(1), a_{n+1}(1)]$ . If  $a_{n+1}(1) > m^2 a_n(1)$ , then

$$\int_0^{m^4 t} \mathbf{E}(x|\mathcal{A}_m)(s) ds \geq \int_0^{ma_n(1)} \mathbf{E}(x|\mathcal{A}_m)(s) ds = X(ma_n(1)) \geq \frac{2}{3}X(t).$$

If  $a_{n+1}(1) \leq m^2 a_n(1)$  and  $a_{n+2}(1) > m^2 a_{n+1}(1)$ , then

$$\int_0^{m^4 t} \mathbf{E}(x|\mathcal{A}_m)(s) ds \geq \int_0^{ma_{n+1}(1)} \mathbf{E}(x|\mathcal{A}_m)(s) ds = X(ma_{n+1}(1)) \geq X(t).$$

If  $a_{n+2}(1) \leq m^2 a_{n+1}(1)$  and  $a_{n+1}(1) \leq m^2 a_n(1)$ , then

$$X(m^4 t) \geq X(a_{n+2}(1)) = \frac{3}{2} X(a_{n+1}(1)) \geq \frac{3}{2} X(t)$$

and the assertion follows.  $\square$

The situation in the case that  $x \in L_1$  is slightly more complicated.

**Lemma 46.** *If  $x = \mu(x) \in L_1(0, 1)$  or  $x \in L_1(0, \infty)$ , then there exists constant  $C$  such that for every  $t > 0$*

$$(26) \quad X(t) \leq \frac{2}{3} X(m^4 t) + \frac{3}{2} \int_0^{m^4 t} \mathbf{E}(x | \mathcal{A}_m)(s) ds + C \int_0^{m^4 t} \chi_{[0,1]}(s) ds.$$

*Proof.* Consider first the case of the semi-axis. Fix  $n_0$  such that  $X(a_{n_0}) \leq 4/9 X(\infty)$ . For a given  $t \in [a, a_{n_0}]$ , there exists  $n \in \mathbb{Z}$  such that  $n < n_0$  and  $t \in [a_n, a_{n+1}]$ . Then, the argument in Lemma 45 applies *mutatis mutandi*. For every  $t \geq a_{n_0}$  we have

$$X(t) \leq \frac{X(\infty)}{\min\{a_{n_0}, 1\}} \min\{m^4 t, 1\} = \frac{X(\infty)}{\min\{a_{n_0}, 1\}} \int_0^{m^4 t} \chi_{[0,1]}(s) ds.$$

Setting  $C = X(\infty)/\min\{a_{n_0}, 1\}$ , we obtain the assertion.

The same argument applies in the case of the interval  $(0, 1)$  by replacing  $X(\infty)$  by  $X(1)$ .  $\square$

The following two theorems are crucial for the proof of the implication (3)  $\Leftrightarrow$  (4) in Theorem 5.

**Theorem 47.** *Let  $E$  be a fully symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis and let  $x \in E$ . Suppose that the norm on  $E$  is a Fatou norm. If  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional on  $E$  and every  $0 \leq y \prec\prec x$ , then*

$$(27) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\|_E = 0$$

provided that one of the following conditions is satisfied

- (1)  $E = E(0, 1)$  is a space on the interval  $(0, 1)$ .
- (2)  $E = E(0, \infty)$  is a space on the semi-axis and  $E(0, \infty) \not\subset L_1(0, \infty)$ .

*Proof.* Without loss of generality,  $x = \mu(x)$ . If  $x \notin L_1$ , then by Lemma 45,

$$\int_0^{t/m^4} x(s) ds \leq \frac{2}{3} \int_0^t x(s) ds + \frac{3}{2} \int_0^t \mathbf{E}(x | \mathcal{A}_m)(s) ds, \quad \forall t > 0$$

or, equivalently,

$$\frac{1}{m^4} \sigma_{m^4} x \prec\prec \frac{2}{3} x + \frac{3}{2} \mathbf{E}(x | \mathcal{A}_m).$$

Applying  $m^{-1} \sigma_m$  to the both parts, we obtain

$$\frac{1}{m^5} \sigma_{m^5} x \prec\prec \frac{2}{3} \frac{1}{m} \sigma_m x + \frac{3}{2} \frac{1}{m} \sigma_m \mathbf{E}(x | \mathcal{A}_m).$$

Take norms and let  $m \rightarrow \infty$ . It follows from Lemma 44 that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m x\|_E \leq \frac{2}{3} \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m x\|_E.$$

This proves (27).

If  $x \in L_1$  and  $C$  are as in Lemma 46, then it follows from Lemma 46 that

$$\int_0^{t/m^4} x(s)ds \leq \frac{2}{3} \int_0^t x(s)ds + \frac{3}{2} \int_0^t \mathbf{E}(x|\mathcal{A}_m)(s)ds + C \int_0^t \chi_{[0,1]}(s)ds, \quad \forall t > 0$$

or, equivalently,

$$\frac{1}{m^4} \sigma_{m^4} x \prec\prec \frac{2}{3} x + \frac{3}{2} \mathbf{E}(x|\mathcal{A}_m) + C \chi_{(0,1)}.$$

Applying  $m^{-1} \sigma_m$  to the both parts, we obtain

$$\frac{1}{m^5} \sigma_{m^5} x \prec\prec \frac{2}{3} \frac{1}{m} \sigma_m x + \frac{3}{2} \frac{1}{m} \sigma_m \mathbf{E}(x|\mathcal{A}_m) + C \frac{1}{m} \sigma_m \chi_{(0,1)}.$$

Take norms and let  $m \rightarrow \infty$ . For every symmetric space  $E$  on the interval  $(0, 1)$  and for every symmetric space  $E$  on the semi-axis such that  $E \not\subset L_1(0, \infty)$  we have  $m^{-1} \|\sigma_m \chi_{(0,1)}\|_E \rightarrow 0$ . It follows from Lemma 44 that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m x\|_E \leq \frac{2}{3} \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m x\|_E$$

and again (27) follows.  $\square$

**Theorem 48.** *Let  $E = E(0, \infty)$  be a fully symmetric Banach space on the semi-axis equipped with a Fatou norm such that  $E(0, \infty) \subset L_1(0, \infty)$ . If  $\varphi(y) \leq \varphi(x)$  for every positive symmetric functional on  $E$  and every  $0 \leq y \prec\prec x$ , then*

$$(28) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E = 0.$$

*Proof.* Fully symmetric Banach space  $F$  on the interval  $(0, 1)$  consists of those  $z \in E$  supported on the interval  $(0, 1)$ . Let  $x_1 = \mu(x)\chi_{(0,1)} \in F$ . Suppose that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x))\chi_{(0,1)}\|_E > 0.$$

It clearly follows that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m(\mu(x_1))\|_F > 0.$$

By Theorem 47, there exists  $0 \leq y_1 \prec\prec x_1$  and a positive symmetric functional  $\varphi \in F^*$  such that  $\varphi(y_1) > \varphi(x_1)$ . Let  $\varphi_{sing}$  be a singular part of the functional  $\varphi$  constructed in Lemma 26. It follows from Lemma 26 that  $\varphi_{sing}$  is symmetric. By Lemma 27, the difference  $\varphi - \varphi_{sing}$  is a symmetric normal functional on  $F$  (that is, an integral). Therefore,  $\varphi_{sing}(y_1) > \varphi_{sing}(x_1)$ .

Now we show that the functional  $\varphi_{sing}$  can be extended from  $F$  to  $E$  by setting

$$\varphi_{sing}(z) = \lim_{n \rightarrow \infty} \varphi_{sing}(\mu(z)\chi_{(0,1/n)}), \quad 0 \leq z \in E.$$

Repeating the argument in Lemma 26, we prove that the extension above is additive on  $E_+$ . Thus, the functional  $\varphi_{sing} \in E^*$  is positive and symmetric. Since  $y_1 \prec\prec x$  and  $\varphi_{sing}(y_1) > \varphi_{sing}(x_1) = \varphi_{sing}(x)$ , the assertion follows.  $\square$

## 7. PROOF OF THEOREM 5

In this section, we prove an assertion more general than that of Theorem 5. The assertion of Theorem 5 follows from that of Theorem 49 by setting  $\mathcal{M} = B(H)$ .

In what follows, the semifinite von Neumann algebra  $\mathcal{M}$  is either atomless or atomic so that the trace of every atom is 1.

**Theorem 49.** *Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space. Consider the following conditions.*

- (1) *There exist nontrivial positive singular symmetric functionals on  $E(\mathcal{M}, \tau)$ .*
- (2) *There exist nontrivial singular fully symmetric functionals on  $E(\mathcal{M}, \tau)$ .*
- (3) *There exist positive symmetric symmetric functional on  $E(\mathcal{M}, \tau)$  which are not fully symmetric.*
- (4) *If  $E(\mathcal{M}, \tau) \not\subset L_1(\mathcal{M}, \tau)$ , then there exists an operator  $A \in E(\mathcal{M}, \tau)$  such that*

$$(29) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \|\sigma_m \mu(A)\|_E > 0.$$

*If  $E(\mathcal{M}, \tau) \subset L_1(\mathcal{M}, \tau)$ , then there exists an operator  $A \in E(\mathcal{M}, \tau)$  such that*

$$(30) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \|(\sigma_m \mu(A))\chi_{(0,1)}\|_E > 0.$$

- (i) *The conditions (1) and (4) are equivalent for every symmetric operator space  $E(\mathcal{M}, \tau)$ .*
- (ii) *The conditions (1), (2) and (4) are equivalent for every fully symmetric operator space  $E(\mathcal{M}, \tau)$ .*
- (iii) *The conditions (1)-(4) are equivalent for every fully symmetric operator space  $E(\mathcal{M}, \tau)$  equipped with a Fatou norm.*

*Proof.* Implications (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1) are trivial.

(1)  $\Rightarrow$  (4) Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space with a singular symmetric functional  $\varphi$ . Let  $A \in E(\mathcal{M}, \tau)$  be an operator such that  $\varphi(A) \neq 0$ . Without loss of generality,  $A \geq 0$ .

If  $E(\mathcal{M}, \tau) \not\subset L_1(\mathcal{M}, \tau)$ , then

$$|\varphi(A)| = \frac{1}{m} |\varphi(\underbrace{A \oplus \cdots \oplus A}_{m \text{ times}})| \leq \|\varphi\|_{E^*(\mathcal{M}, \tau)} \cdot \frac{1}{m} \|\sigma_m \mu(A)\|_E.$$

Passing  $m \rightarrow \infty$ , we obtain the required inequality (29).

Let now  $E(\mathcal{M}, \tau) \subset L_1(\mathcal{M}, \tau)$ . If  $\mathcal{M}$  is atomic, then  $E(\mathcal{M}, \tau) = L_1(\mathcal{M}, \tau)$  and the assertion is trivial. Let  $\mathcal{M}$  be atomless. Since  $\varphi$  is a singular functional and

$$A - AE_A(\mu(\frac{1}{m}, A), \infty) \in (L_1 \cap L_\infty)(\mathcal{M}, \tau), \quad \forall m \in \mathbb{N},$$

we infer that

$$\begin{aligned} |\varphi(A)| &= |\varphi(AE_A(\mu(\frac{1}{m}, A), \infty))| = \\ &= \frac{1}{m} |\varphi(\underbrace{AE_A(\mu(\frac{1}{m}, A), \infty) \oplus \cdots \oplus AE_A(\mu(\frac{1}{m}, A), \infty)}_{m \text{ times}})| \leq \\ &\leq \|\varphi\|_{E^*(\mathcal{M}, \tau)} \cdot \frac{1}{m} \|\sigma_m \mu(AE_A(\mu(\frac{1}{m}, A), \infty))\|_E \leq \|\varphi\|_{E^*(\mathcal{M}, \tau)} \cdot \frac{1}{m} \|(\sigma_m \mu(A))\chi_{(0,1)}\|_E. \end{aligned}$$



Passing  $m \rightarrow \infty$ , we obtain the required inequality (30).

(4)  $\Rightarrow$  (1) Firstly, we assume that the algebra  $\mathcal{M}$  is finite. Without loss of generality,  $\tau(1) = 1$ . Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space and let  $E(0, 1)$  be the corresponding symmetric function space. By the assumption, there exists an element  $x = \mu(A) \in E(0, 1)$  such that  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $E(0, 1)$ . By Theorem 29, there exists a positive singular symmetric functional  $0 \neq \varphi \in E(0, 1)^*$ . Let  $\mathcal{L}(\varphi)$  be a functional on  $E(\mathcal{M}, \tau)$  defined in Theorem 14. Clearly,  $\mathcal{L}(\varphi)$  is a nontrivial positive symmetric functional on  $E(\mathcal{M}, \tau)$ .

The case when  $\mathcal{M}$  is an infinite atomless von Neumann algebra can be treated in a similar manner. The only difference is that the reference to Theorem 29 has to be replaced with the reference to either Theorem 28 or Theorem 23.

Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space on a atomic von Neumann algebra  $\mathcal{M}$  and let  $E(\mathbb{N})$  be the corresponding symmetric sequence space. It follows from the assumption that  $E(\mathcal{M}, \tau) \neq L_1(\mathcal{M}, \tau)$  or, equivalently,  $E(\mathbb{N}) \neq l_1$ . By the assumption, there exists an element  $x = \mu(A) \in E$  such that  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $E$ . Let  $F(0, \infty)$  be a symmetric function space constructed in Proposition 16. Since  $E(\mathbb{N}) \neq l_1$ , it follows that  $F(0, \infty) \not\subset L_1(0, \infty)$ . Recall that the space  $E(\mathbb{N})$  is naturally embedded into the space  $F(0, \infty)$  and that the norms  $\|\cdot\|_E$  and  $\|\cdot\|_F$  are equivalent on  $E(\mathbb{N})$ . We have  $x \in F$  and  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $F(0, \infty)$ . By Theorem 23, there exists a positive symmetric functional  $0 \leq \varphi \in F(0, \infty)^*$ . The restriction of the functional  $\varphi$  to  $E(\mathbb{N})$  is a nontrivial positive symmetric functional on  $E(\mathbb{N})$ . Let  $\mathcal{L}(\varphi)$  be a functional on  $E(\mathcal{M}, \tau)$  defined in Theorem 14. Clearly,  $\mathcal{L}(\varphi)$  is a nontrivial positive symmetric functional on  $E(\mathcal{M}, \tau)$ .

(4)  $\Rightarrow$  (2) The proof is very similar to that of the implication (4)  $\Rightarrow$  (1) and is, therefore, omitted. The only difference is that references to Theorem 29, Theorem 28 or Theorem 23 have to be replaced with references to Theorem 36, Theorem 35 or Theorem 33, respectively.

(4)  $\Rightarrow$  (3) Firstly, we assume that the algebra  $\mathcal{M}$  is finite. Without loss of generality,  $\tau(1) = 1$ . Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space and let  $E(0, 1)$  be the corresponding symmetric function space. By the assumption, there exists an element  $x = \mu(A) \in E(0, 1)$  such that  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $E(0, 1)$ . By Theorem 47, there exists a positive symmetric but not fully symmetric functional  $\varphi \in E(0, 1)^*$ . Let  $\mathcal{L}(\varphi)$  be a functional on  $E(\mathcal{M}, \tau)$  defined in Theorem 14. Clearly,  $\mathcal{L}(\varphi)$  is a symmetric but not fully symmetric functional on  $E(\mathcal{M}, \tau)$ .

The case when  $\mathcal{M}$  is an infinite atomless von Neumann algebra can be treated in a similar manner. The only difference is that the reference to Theorem 47 has to be replaced with the reference to either Theorem 47 or Theorem 48.

Let  $E(\mathcal{M}, \tau)$  be a symmetric operator space on a atomic von Neumann algebra  $\mathcal{M}$  and let  $E(\mathbb{N})$  be the corresponding symmetric sequence space. It follows from the assumption that  $E(\mathcal{M}, \tau) \neq L_1(\mathcal{M}, \tau)$  or, equivalently,  $E(\mathbb{N}) \neq l_1$ . By the assumption, there exists an element  $x = \mu(A) \in E$  such that  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $E$ . Let  $F(0, \infty)$  be a symmetric function space constructed in Proposition 16. Since  $E(\mathbb{N}) \neq l_1$ , it follows that  $F(0, \infty) \not\subset L_1(0, \infty)$ . Recall that the space  $E(\mathbb{N})$  is naturally embedded into the space  $F(0, \infty)$  and that the norms  $\|\cdot\|_E$  and  $\|\cdot\|_F$  are equivalent on  $E(\mathbb{N})$ . We have  $x \in F$  and  $m^{-1}\sigma_m x \not\rightarrow 0$  in  $F(0, \infty)$ . By Theorem 47, there exists a positive symmetric functional  $\varphi \in F(0, \infty)^*$  and a function  $0 \leq y \prec\prec x$  such that  $\varphi(y) > \varphi(x)$ . Set  $z = \mathbf{E}(\mu(y)|\{(n-1, n)\}_{n \in \mathbb{N}})$ . Clearly,  $z \in E(\mathbb{N})$  and  $\varphi(z) = \varphi(y) > \varphi(x)$ . Hence, the restriction of the functional  $\varphi$  to  $E(\mathbb{N})$  is a positive

symmetric but not fully symmetric functional on  $E(\mathbb{N})$ . Let  $\mathcal{L}(\varphi)$  be a functional on  $E(\mathcal{M}, \tau)$  defined in Theorem 14. Clearly,  $\mathcal{L}(\varphi)$  is a positive symmetric but not fully symmetric functional on  $E(\mathcal{M}, \tau)$ .  $\square$

## 8. APPENDIX

In this appendix, we set  $\mathcal{A} = \{(n-1, n)\}_{n \in \mathbb{N}}$ .

**Lemma 50.** *If  $x, y \in (L_1 + L_\infty)(0, \infty)$  are positive functions, then*

$$\mathbf{E}(\mu(x+y)|\mathcal{A}) \triangleleft \mathbf{E}(\mu(x)|\mathcal{A}) + \mathbf{E}(\mu(y)|\mathcal{A}) \triangleleft 2\sigma_{1/2}\mathbf{E}(\mu(x+y)|\mathcal{A}).$$

*Proof.* Recall that

$$\mu(x+y) \prec \mu(x) + \mu(y) \prec 2\sigma_{1/2}\mu(x+y).$$

It follows that

$$\begin{aligned} \int_0^b \mu(s, x+y) ds &\leq \int_0^b (\mu(s, x) + \mu(s, y)) ds, \\ \int_0^{2a} \mu(s, x+y) ds &\geq \int_0^a (\mu(s, x) + \mu(s, y)) ds. \end{aligned}$$

Let now  $a, b$  be positive integers. Subtracting the above inequalities, we obtain

$$\begin{aligned} \int_{2a}^b \mathbf{E}(\mu(x+y)|\mathcal{A})(s) ds &= \int_{2a}^b \mu(s, x+y) ds \leq \\ &\leq \int_a^b (\mu(s, x) + \mu(s, y)) ds = \int_a^b \mathbf{E}(\mu(x) + \mu(y)|\mathcal{A})(s) ds. \end{aligned}$$

Similarly, we have

$$\int_{2a}^b \mathbf{E}(\mu(x) + \mu(y)|\mathcal{A})(s) ds \leq \int_{2a}^{2b} \mathbf{E}(\mu(x+y)|\mathcal{A})(s) ds.$$

$\square$

**Corollary 51.** *The quasi-norm in Construction 16 is a norm.*

*Proof.* It follows from Lemma 50 that

$$\mathbf{E}(\mu(x+y)|\mathcal{A}) \triangleleft \mathbf{E}(\mu(x) + \mu(y)|\mathcal{A})$$

provided that  $x, y$  are positive functions. By Theorem 9,

$$\|\mathbf{E}(\mu(x+y)|\mathcal{A})\|_E \leq \|\mathbf{E}(\mu(x)|\mathcal{A})\|_E + \|\mathbf{E}(\mu(y)|\mathcal{A})\|_E.$$

$\square$

**Lemma 52.** *Let  $y = \mu(y) \in (L_1 + L_\infty)(0, \infty)$ . It follows that*

$$\int_{2^{-k}\lambda a}^b y(s) ds \leq \frac{\lambda}{\lambda-1} \int_a^b (\sigma_{2^k} y)(s) ds$$

*provided that  $b \geq \lambda a$ .*

*Proof.* Let  $\alpha$  be the average value of  $y$  on the interval  $[2^{-k}\lambda a, 2^{-k}b]$ . Clearly,  $y \leq \alpha$  on the interval  $[2^{-k}\lambda a, b]$  and  $y \geq \alpha$  on the interval  $[2^{-k}a, 2^{-k}b]$ . Thus,  $\sigma_{2^k} y \geq \alpha$  on the interval  $[a, b]$ . Therefore,

$$\int_{2^{-k}\lambda a}^b y(s) ds \leq (b - 2^{-k}\lambda a)\alpha \leq \frac{\lambda}{\lambda-1}(b-a)\alpha \leq \frac{\lambda}{\lambda-1} \int_a^b (\sigma_{2^k} y)(s) ds.$$

$\square$

**Theorem 53.** *If  $\{x_n\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $F$ , then there exists  $x \in F$  such that  $x_n \rightarrow x$  in  $F$ .*

*Proof.* For every  $k > 0$ , there exists  $m_k$  such that  $\|x_m - x_{m_k}\|_F \leq 4^{-k}$  for  $m \geq m_k$ . Set  $y_k = x_{m_{k+1}} - x_{m_k}$ . Clearly,  $\|y_k\|_F \leq 4^{-k}$  for every  $k \in \mathbb{N}$ . In particular, the series  $\sum_{k=1}^{\infty} y_k$  converges in  $L_{\infty}(0, \infty)$ .

Set  $z_n = \sum_{k=n}^{\infty} \sigma_{2^k} \mu(y_k)$ . We claim that  $z_n \in F$  and  $z_n \rightarrow 0$  in  $F$ . Indeed,

$$\mu(y_k) \leq \|y_k\|_{\infty} \chi_{(0,1)} + T\mathbf{E}(\mu(y_k)|\mathcal{A}).$$

Here,  $T$  is a shift to the right. It follows that

$$\mathbf{E}(\mu(z_n)|\mathcal{A}) \leq \sum_{k=n}^{\infty} \sigma_{2^k} (\|y_k\|_{\infty} \chi_{(0,1)} + T\mathbf{E}(\mu(y_k)|\mathcal{A})).$$

Therefore,

$$\begin{aligned} \|z_n\|_F &\leq \|z_n\|_{\infty} + \sum_{k=n}^{\infty} 2^k \|\|y_k\|_{\infty} \chi_{(0,1)} + T\mathbf{E}(\mu(y_k)|\mathcal{A})\|_E \leq \\ &\leq \|z_n\|_{\infty} + \sum_{k=n}^{\infty} 2^{k+1} \|y_k\|_F \leq \frac{1}{3} \cdot 4^{1-n} + 2^{2-n} = o(1). \end{aligned}$$

It follows from Lemma 8.5 of [19] that

$$\int_{\lambda a}^b \mu(s, \sum_{k=n}^{\infty} y_k) ds \leq \sum_{k=n}^{\infty} \int_{2^{-k} \lambda a}^b \mu(s, y_k) ds.$$

It follows from Lemma 52 that

$$\int_{2^{-k} \lambda a}^b \mu(s, y_k) ds \leq \frac{\lambda}{\lambda - 1} \int_a^b (\sigma_{2^k} \mu(y_k))(s) ds.$$

Therefore,

$$\int_{\lambda a}^b \mu(s, \sum_{k=n}^{\infty} y_k) ds \leq \frac{\lambda}{\lambda - 1} \int_a^b z_n(s) ds.$$

Hence,

$$\sum_{k=n}^{\infty} y_k \triangleleft \frac{\lambda}{\lambda - 1} z_n.$$

Since  $\lambda > 1$  is arbitrarily large, it follows from Theorem 9 that

$$\|\sum_{k=n}^{\infty} y_k\|_F \leq \|z_n\|_F \rightarrow 0.$$

Thus, the series  $\sum_{k=1}^{\infty} y_k$  does converge in  $F$ . The assertion follows immediately.  $\square$

## 9. PROOF OF FIGIEL-KALTON THEOREM

The proof of Theorem 8 follows from the combinations of Lemmas below.

**Lemma 54.** *Let  $E$  be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. If  $x \in Z_E$ , then  $C(\mu(x_+) - \mu(x_-)) \in E$ .*

*Proof.* Let  $x = \sum_{k=1}^n (x_k - y_k)$  with  $x_k, y_k \in E_+$  and  $\mu(x_k) = \mu(y_k)$ ,  $1 \leq k \leq n$ . Set

$$z = x_+ + \sum_{k=1}^n y_k = x_- + \sum_{k=1}^n x_k.$$

It follows from the definition of  $C$  and (8) that

$$C\mu(z) \leq C(x_+) + \sum_{k=1}^n C\mu(y_k) = C(\mu(x_+) - \mu(x_-)) + C\mu(x_-) + \sum_{k=1}^n C\mu(x_k).$$

Using the second inequality in (8), we obtain

$$\int_0^t (\mu(s, x_-) + \sum_{k=1}^n \mu(s, x_k)) ds \leq \int_0^{(n+1)t} \mu(s, z) ds \leq \int_0^t \mu(s, z) ds + nt\mu(t, z).$$

Therefore,

$$C\mu(z) \leq C\mu(z) + C(\mu(x_+) - \mu(x_-)) + n\mu(z).$$

It follows that  $C(\mu(x_-) - \mu(x_+)) \leq n\mu(z)$ . Similarly,  $C(\mu(x_+) - \mu(x_-)) \leq n\mu(z)$  and the assertion follows.  $\square$

**Lemma 55.** *Let  $E$  be a symmetric Banach space either on the interval  $(0, 1)$  or on the semi-axis. If  $x \in \mathcal{D}_E$ , then  $C(\mu(x_+) - \mu(x_-)) \in Cx + E$ .*

*Proof.* Since  $x \in \mathcal{D}_E$ , it follows that  $x = \mu(a) - \mu(b)$  with  $a, b \in E$ . Set  $u = \mu(a) - x_+ \geq 0$ . Clearly,  $\mu(a) = u + x_+$  and  $\mu(b) = u + x_-$ . It follows from the definition of  $C$  and (8) that

$$C\mu(a) \leq C\mu(u) + C\mu(x_+) = C(\mu(x_+) - \mu(x_-)) + C\mu(u) + C\mu(x_-).$$

Using the second inequality in (8), we obtain

$$C\mu(x_-) + C\mu(u) \leq C\mu(b) + \mu(b).$$

It follows that

$$Cx \leq C(\mu(x_+) - \mu(x_-)) + \mu(b).$$

Similarly,

$$Cx \geq C(\mu(x_+) - \mu(x_-)) - \mu(a)$$

and the assertion follows.  $\square$

**Lemma 56.** *Let  $E = E(0, \infty)$  be a symmetric space on the semi-axis. If  $x \in \mathcal{D}_E$  is such that  $Cx \in E$ , then  $x \in Z_E$ .*

*Proof.* Define a partition  $\mathcal{A} = \{(2^n, 2^{n+1})\}_{n \in \mathbb{Z}}$  and set  $x_1 = \mathbf{E}(x|\mathcal{A})$ . If  $x = \mu(a) - \mu(b)$  with  $a, b \in E$ , then  $x_1 = \mathbf{E}(\mu(a)|\mathcal{A}) - \mathbf{E}(\mu(b)|\mathcal{A})$ . Clearly,

$$\mathbf{E}(\mu(a)|\mathcal{A}) \leq \sigma_2 \mu(a) \in E, \quad \mathbf{E}(\mu(b)|\mathcal{A}) \leq \sigma_2 \mu(b) \in E$$

are decreasing functions. It follows that  $x_1 \in \mathcal{D}_E$ . It is easy to see that

$$|Cx_1 - Cx| \leq 2\sigma_2(\mu(a) + \mu(b)).$$

Therefore,  $Cx_1 \in E$ . Define a function  $z \in E$  by setting

$$z(t) = (Cx_1)(2^{n+1}), \quad t \in (2^n, 2^{n+1}).$$

Clearly,  $x_1 = 2z - \sigma_2 z \in Z_E$ .

Consider the function  $x - x_1$  on the interval  $(2^n, 2^{n+1})$ . By Kwapien theorem [23], there exist positive equimeasurable functions  $y_{1n}, y_{2n}$  supported on  $(2^n, 2^{n+1})$  such that

$$\mu(y_{1n}) = \mu(y_{2n}), \quad \|y_{1n}\|_\infty, \|y_{2n}\|_\infty \leq 6\|(x - x_1)\chi_{(2^n, 2^{n+1})}\|_\infty.$$

Set  $y_1 = \sum_{n \in \mathbb{N}} y_{1n}$  and  $y_2 = \sum_{n \in \mathbb{N}} y_{2n}$ . It follows that  $y_1, y_2 \in E_+$ . Since  $x - x_1 = y_1 - y_2$  and  $\mu(y_1) = \mu(y_2)$ , it follows that  $x - x_1 \in Z_E$ . The assertion follows immediately.  $\square$

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